Mathematical Techniques: Part 1. Complex Algebra

Lecture notes by W.B. von Schlippe

Complex numbers are an extension of real numbers. The need for complex numbers arises when we must take square roots of negative numbers. Thus there is no real number whose square is equal to -1. We therefore define a new entity called the *imaginary unit* i:

$$i = \sqrt{-1} \tag{1}$$

(Note that it is common practice of physicists and mathematicians to use the letter i for the imaginary unit; in electrical engineering the letter j is preferred.)

It follows immediately from the definition of i that

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = i, \quad i^5 = -1, \ldots \tag{2}$$

The square root of an arbitrary negative number $-a$, $a > 0$, follows then simply as

$$\sqrt{-a} = \sqrt{(-1)}\sqrt{a} = i\sqrt{a} \tag{3}$$

i.e. it is i times the square root of the absolute value of $-a$.

A number such as i itself or as ia, where a is a real number, is called an *imaginary* number. A *complex* number is the sum of a real number and an imaginary number. Thus if a and b are two real numbers, then c, defined by

$$a + ib,$$

is a complex number. a is the *real part* of c and b is the *imaginary part* of c. In physics it is usual to denote the real part of a complex number c by $\mathrm{Re}\, c$ and the imaginary part by $\mathrm{Im}\, c$. Thus in the above example we have

$$a = \mathrm{Re}\, c \qquad \text{and} \qquad b = \mathrm{Im}\, c \tag{4}$$

An important property of real numbers is that they are *ordered*. Thus for any pair of real numbers a and b one of the relations

$$a > b \qquad \text{or} \qquad a = b \qquad \text{or} \qquad a < b$$

is true. It is important to realize that complex numbers are *not* ordered in the above sense. In other words, it is meaningless to say that one complex number is greater or less than another complex number. Only the *equality* of complex numbers is defined: given two complex numbers $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, then $c_1 = c_2$ iff[1] $a_1 = a_2$ and $b_1 = b_2$.

Addition of complex numbers.

The sum of two complex numbers $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, where a_1, a_2, b_1 and b_2 are real numbers, is a complex number $c = c_1 + c_2$ whose real part is $a_1 + a_2$ and whose imaginary part is $b_1 + b_2$:

$$c = c_1 + c_2 = (a_1 + a_2) + i(b_1 + b_2) \tag{5}$$

[1] we shall use the shorthand iff to mean *if and only if*

Multiplication of a complex number by a real number.
Given a complex number $c = a + ib$ and a real number λ, then the product λc is a complex number whose real part is λa and whose imaginary part is λb:

$$\lambda c = \lambda(a + ib) = \lambda a + i(\lambda b) \tag{6}$$

In particular, if $\lambda = -1$, then

$$-c = -a - ib \tag{7}$$

Subtraction of two complex numbers.
The difference of two complex numbers $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, in that order, is the complex number c, which is obtained as the *sum* of c_1 and $-c_2$:

$$c = c_1 - c_2 = c_1 + (-c_2) \tag{8}$$

This means that the subtraction does not need a new definition but is reduced to addition and multiplication by a real number.

Multiplication of complex numbers.
The multiplication of complex numbers follows the rules of ordinary algebra. Given two complex numbers, $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, their product is a complex number c which we get by applying the usual rules of algebra, only remembering that $i^2 = -1$. First we multiply out the brackets, treating the imaginary unit i like any other number:

$$c = c_1 c_2 = (a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 + (ib_1)(ib_2) + a_1(ib_2) + (ib_1)a_2 \tag{9}$$

then we note that in the second term there are two factors of i, so we can write $(ib_1)(ib_2) = i^2 b_1 b_2 = -b_1 b_2$, where in the last step the fundamental property of the imaginary unit was used. The third and the fourth terms each have one factor of i which can be put outside a common bracket. Thus we get finally

$$c = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + b_1 a_2) \tag{10}$$

Examples.
1.) Find the sum c of the complex numbers $c_1 = 3 + 2i$ and $c_2 = -1 + 7i$.
Answer: $c = 2 + 9i$.
2.) Find the difference $c = c_1 - c_2$ if $c_1 = 5 - 3i$ and $c_2 = 2 + 7i$.
Answer: $c = 3 - 10i$.
3.) Find the product $c = c_1 c_2$ of the complex numbers $c_1 = 4 + 3i$ and $c_2 = 1 + 8i$.
Answer: $c = -20 + 35i$.
4.) Find the product $c = c_1 c_2$ of the complex numbers $c_1 = 1 + i$ and $c_2 = 1 - i$.
Answer: $c = 2$.

You should certainly spend enough time over these examples to let them sink in. But then you will find that only the last example has an unexpected result: did we not expect to get a complex number? Instead we did get a real number. The reason for this is the simple relationship between the two complex numbers: their real parts are equal and their imaginary parts are of equal magnitude but have opposite signs. We can convince ourselves that the product of any pair of complex numbers with such property is a real number.

Consider the complex numbers $c_1 = a + ib$ and $c_2 = a - ib$. I have chosen them to have the property described above: their real parts are equal and their imaginary parts are of equal magnitude but have opposite signs. Now take their product using the rule from above:

$$c = c_1 c_2 = (a + ib)(a - ib) = a^2 + b^2 + i(ba - ab) = a^2 + b^2 \tag{11}$$

and the result is a real number for any real a and b.

Because of the special nature of the relationship between complex numbers such as $a + ib$ and $a - ib$ they have a special name: they are called the *complex conjugate* of each other. Thus $(a-ib)$ is the complex conjugate of $(a+ib)$, but of course also $(a+ib)$ is the complex conjugate of $(a-ib)$. There is also a special symbol that goes with this property, except that here physicists and mathematicians use on occasions different symbols: in physics it is usual to denote the complex conjugate of c by c^*, whereas mathematicians frequently use $\bar{c}$ to denote the complex conjugate.[2] One should be aware of such differences in notation to avoid being disoriented when reading an unfamiliar text. In these notes I shall consistently use the physicists notation to denote complex conjugation:

$$(a + ib)^* = a - ib, \qquad (a - ib)^* = a + ib \tag{12}$$

Because of the great importance of the expression $(a^2 + b^2)$ in relation to the complex number $(a + ib)$ it has also a special name: it is called the *modulus squared* of $(a + ib)$, and its *positive* square root is known as the *modulus*. There is also a special symbol, this time used universally by physicists, mathematicians and electricians: one writes for the modulus squared of $c = a + ib$:

$$|c|^2 = |a + ib|^2 = a^2 + b^2 \tag{13}$$

and correspondingly for its modulus

$$|c| = |a + ib| = \sqrt{a^2 + b^2} \tag{14}$$

and it is understood that the modulus is a positive quantity.

Examples.
5.) Find the complex conjugate of the complex number $(8 - 37i)$.
Answer: $8 + 37i$.
6.) Find the modulus of the complex number $(3 + 4i)$.
Answer: 5.
7.) Find the modulus squared of the complex number $(-1 - 7i)$.
Answer: 50.

Division of complex numbers.
The division of complex numbers does not require a new definition but can be reduced to the rule of multiplication. Indeed, consider the quotient of two complex numbers $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$:

$$c = \frac{c_1}{c_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \tag{15}$$

[2] Mathematicians also have the habit of writing c' for the real part of c and c'' for the imaginary part.

if we now multiply both the numerator and the denominator by c_2^*, then we get a *real* number in the denominator, namely the modulus squared of c_2; thus we proceed to get

$$c = \frac{c_1 c_2^*}{|c_2|^2} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{a_2^2 + b_2^2} \tag{16}$$

and we have succeeded in reducing the division of c_1 by c_2 to the multiplication of c_1 with c_2^*, followed by the multiplication by the real number $1/(a^2 + b^2)$. Let us finish the calculation and get the final result in the form

$$c = \frac{c_1}{c_2} = \frac{a_1 a_2 + b_1 b_2 + i(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2} \tag{17}$$

Examples.
8.) Find the quotient of the division of $c_1 = 2 + 3i$ by $c_2 = 4 - 5i$.
Answer: $-\frac{7}{41} + i\frac{22}{41}$.
9.) Find the quotient of the division of $c_1 = 7 + 6i$ by $c_2 = 14 - 12i$.
Answer: $\frac{13}{170} + i\frac{84}{170}$.
10.) Find the quotient of the division of $c_1 = 1 + i$ by $c_2 = 1 - i$.
Answer: i.

Let us consider Examples 8 - 10 carefully. We can note that examples 9 and 10 have something in common: in example 10 the denominator is the complex conjugate of the numerator, and in example 9 the denominator is also the complex conjugate of the numerator, but only up to a factor of two. Yet their answers are strikingly different: in example 10 the answer is an imaginary number, i.e. the real part of the quotient is zero, whereas in example 10 both real and imaginary parts are nonzero. So, what is special about example 10? Well, the real and imaginary parts are of equal magnitude, unlike the case of example 9, where they are different. Therefore let us ask ourselves the question: is it a general rule that the quotient of a complex number and its complex conjugate is an imaginary number if the real and imaginary parts are equal in magnitude? The answer is easily found because any complex number whose real and imaginary parts are equal in magnitude can be written in the form of $c = a(1 + i)$ (or $c = a(1 - i)$), where a is a real number. Therefore the quotient in question is

$$\frac{c}{c^*} = \frac{a(1+i)}{a(1-i)} = \frac{1+i}{1-i}$$

i.e. it is reduced to the case of example 10. Similarly if we have $c = a(1 - i)$ then we get

$$\frac{c}{c^*} = \frac{a(1-i)}{a(1+i)} = \frac{1-i}{1+i}$$

and it is easy to show that in this case the answer is $-i$.

Argand diagram

Many aspects of complex algebra become clearer in graphical form. One draws a Cartesian set of coordinate axes and plots the real parts of complex numbers along the abscissa and their imaginary parts along the ordinate. The plane spanned by these axes is called the *complex plane*, and the representation of complex numbers in this plane is known as an *Argand diagram.*

The imaginary part of a complex number is of course a real number, and on paper we can plot only real values, but it is customary to label the unit on the ordinate of the complex plane with i rather than with 1.

Each complex number is represented by a point in the complex plane. The coordinates of the point are its real and imaginary parts, respectively. Frequently it is convenient to join the point, representing a complex number, to the origin of the complex plane, marking the end point with an arrow, like a two-dimensional vector. This convention is the basis for the rules of graphical constructions of the elementary operations, addition, subtraction etc.,

To add the complex numbers $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$ graphically we put them on the complex plane and construct the two corresponding vectors. We also mark their real and imaginary parts on the axes. To add the numbers we must add their real and imaginary parts separately. To add the real parts we mark the distance a_2 from the point a_1 on the abscissa. Similarly we proceed to add the imaginary parts, and finally we construct the point corresponding to the sum $c_1 + c_2$. Having made this construction we will notice that our procedure was equivalent to sliding the vector c_2 parallel to itself with its starting point along the vector c_1 all the way to the tip of the arrow of c_1. The two vectors are then said to be *chained.* The vector representing the sum $c = c_1 + c_2$ is then the third side of a triangle formed by the vectors c_1, c_2 and c.

To subtract the complex number c_2 from c_1 graphically we follow the prescription of the algebraic method: we begin by constructing the vectors representing c_1 and c_2, then construct the vector $-c_2$, i.e. the vector whose real part is $-a_2$ and imaginary part is $-b_2$, and then we add the vectors c_1 and $-c_2$.

Polar representation of complex numbers

Somewhat more involved than addition and subtraction in the Argand diagram are multiplication and division. Before we attempt to do that it will be useful to introduce yet another new concept, the *polar form* of complex numbers.

The polar form of representation of complex numbers is actually suggested already by their representation in the complex plane. Indeed, any point P on the Argand diagram can be described either by its cartesian coordinates or by its polar coordinates, i.e. by the distance r from the origin O and the angle θ which the vector OP makes with the positive real axis. As usual we use the convention that angles, measured in anticlockwise direction are positive, angles measured in clockwise direction are negative.

With this definition we can immediately get a relation between the polar and the cartesian representations of complex numbers.

Consider the complex number $c = a + ib$. Denote its polar coordinates by r and θ. From the Argand diagram we can read off that

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

and hence

$$c = a + ib = r(\cos\theta + i\sin\theta) \tag{18}$$

We also note that r is the modulus of c. Indeed

$$|c| = \sqrt{a^2 + b^2} = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = r$$

because of the identity $\cos^2\theta + \sin^2\theta = 1$. The angle θ also has a special name: it is called the *argument* of c, and one writes[3] $\theta = \arg c$. By inspection of Eq. (18) we can find the expression of the argument of c in terms of its real and imaginary parts:

$$\theta = \arctan \frac{\operatorname{Im} c}{\operatorname{Re} c}$$

More convenient than writing the polar form of a complex number in terms of trigonometric functions is the equivalent form

$$c = re^{i\theta}$$

To convince ourselves that this is identical with the polar form in terms of trigonometric functions we can use the Taylor expansions of cos, sin and exp. Let us begin by writing down the expansion of $\exp(i\theta)$:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots + \frac{(i\theta)^n}{n!} + \ldots$$

and recall that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc. Thus collecting the real and imaginary terms separately we can write

$$\begin{aligned} e^{i\theta} &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - + \ldots \\ &+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \ldots\right) \end{aligned} \tag{19}$$

and we recognize in the series of the real terms the Taylor expansion of $\cos\theta$ and in the series of the imaginary terms the expansion of $\sin\theta$.

The exponential representation of complex numbers is particularly well suited for the multiplication of complex numbers. Consider two complex numbers $c_1 = r_1 \exp(i\theta_1)$ and $c_2 = r_2 \exp(i\theta_2)$. Their product is

$$c = c_1 c_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

i.e. the moduli of the complex numbers multiply and their arguments add. Now, the multiplication of the moduli is not easily done graphically, but there is no need to do that, since it is easy enough to multiply them arithmetically. On the other hand, the graphical addition of the arguments is so obvious that it hardly needs an explanation.

To divide c_1 by c_2 using the exponential notation is also easy:

$$c = \frac{c_1}{c_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

i.e. the modulus of c is the quotient of the moduli of c_1 and c_2 and its argument is the difference of their arguments.

A special case of multiplication is the raising to a power. Consider the raising of $c = r \exp i\theta$ to the nth power:

$$c^n = (re^{i\theta})^n = r^n e^{in\theta}$$

[3]another notation occasionally found in the literature is arc c, where arc derives from the Latin "arcus" with obvious meaning.

i.e. the modulus of c is raised to the nth power and its argument is multiplied by n.

We get an interesting result if we rewrite the last equation in terms of trigonometric functions in the particular case of $r = 1$:

$$c^n = e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

This identity is known as *de Moivre's theorem.*

If we carry out the binomial expansions of $(\cos\theta + i\sin\theta)^n$ for $n = 2, 3, 4, \ldots$ and equate the real and imaginary parts on the two sides of the identity, we can get expressions for the cosines and sines of multiples of θ in terms of powers of $\cos\theta$ and $\sin\theta$. For instance,

$$c^2 = (\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

and after opening the bracket and equating the real and imaginary parts, we get the following formulas, familiar from elementary trigonometry:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad \text{and} \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

Similarly we can get

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \qquad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^2\theta$$

In our discussion of the raising of a complex number to a power it was not essential that n was an integer. All our formulæ remain valid for arbitrary real values of n. Of particular interest are cases such as $c^{1/2}$ or $c^{1/3}$ etc., i.e. the square root and cube root etc. of c. Thus for $c = r\exp(i\theta)$ we get

$$c^{1/2} = \sqrt{r}e^{i\frac{\theta}{2}}$$

Now this may look simple enough, but there is a slight subtlety here, and that is connected with the periodicity of the trigonometric functions. Indeed, the sin and cos of an angle do not change if we add a multiple of 2π to that angle:

$$\cos(\theta + 2k\pi) = \cos\theta, \qquad \sin(\theta + 2k\pi) = \sin\theta.$$

where k is an integer. This ambiguity is frequently unimportant, but it is *very* important when we take roots of complex numbers. In the above example we could have written c in the form of $c = r\exp[i(\theta + 2k\pi)]$, and then the nth root would become

$$c^{1/n} = \sqrt[n]{r}\exp\left(i\frac{\theta + 2k\pi}{n}\right)$$

and we get n different complex numbers for $k = 0, 1, 2, \ldots, (n-1)$. The complex number corresponding to $k = n$ coincides with that for $k = 0$, so this is not a new root, and repeating the argument for $k > n$ we convince ourselves that there are indeed exactly n roots.

Examples.

11.) Use graph paper and a ruler to construct the Argand diagrams of the following operations on complex numbers:

(a) add the numbers $0.75 + 0.5i$ and $1.25 + i$,

(b) subtract the number $1.5 + 3i$ from $-0.5 + i$,

(c) multiply the numbers $1.2\exp(i\pi/6)$ and $0.9\exp(i\pi/4)$,
(d) divide $3\exp(i\pi/3)$ by $4\exp(i\pi/2)$.
12.) Find all cube roots of 1 and plot them on an Argand diagram;
[Hint: write 1 as $\exp(2k\pi i)$, $k = 0, 1, 2, \ldots$].
13.) Find all fourth roots of -1 and plot them on an Argand diagram.

Exercise 1. Show that $\arg\frac{1}{c} = \arg c^* = -\arg c$.

Applications

The applications of complex numbers in physics are numerous. Already in your 1st year courses you have been introduced to electric circuits, electromagnetic waves, matter waves, and in all these cases the use of complex numbers is either a great simplification of the theory or indispensable. In quantum mechanics, for instance, there is no way of avoiding complex numbers simply because wave functions, describing matter waves, are complex quantities except in some particular cases. In optics it is convenient to combine the refractive index and the absorption coefficient into one complex quantity, also called the (complex) refractive index, such that its imaginary part is the absorption.

In this course we shall discuss one case in some detail, namely the theory of a.c circuits. This is closely connected with the mathematical theory of 2nd order differential equations with constant coefficients.

Application to a.c. circuits[4]

In the analysis of a.c. circuits one has to consider networks whose elements are resistors, capacitors and induction coils. These are characterized by their resistance R, capacitance C and inductance L, respectively. In the simplest case of a sinusoidal oscillation with circular frequency[5] ω there are three fundamental quantities with the dimension of a resistance: the resistance R itself, and the *reactances* $1/\omega C$ and ωL. Current and voltage are treated as complex quantities:

$$I = I_0 e^{j\omega t}, \qquad V = V_0 e^{j(\omega t + \delta)}$$

where i_0 and V_0 are real. The generalization of Ohm's law to a.c. circuits is of the form

$$V = IZ$$

where Z is called the *impedance*. The real part of Z is the resistance and the imaginary part of Z is the *reactance* X:

$$R = \mathrm{Re}\, Z, \qquad X = \mathrm{Im}\, Z$$

i.e.

$$Z = R + jX$$

With these definitions the rules of combining resistances in d.c. circuits are immediately translated into similar rules for the impedances:

$$\text{impedances } Z_1, Z_2, \ldots, Z_n \text{ in series:} \qquad Z = Z_1 + Z_2 + \ldots + Z_n$$

[4]in this section, as a concession to the custom in electrical engineering, the complex unit will be denoted by j, thus $j = \sqrt{-1}$.

[5]the *circular frequency* ω is related to the *frequency* f by $\omega = 2\pi f$. We shall always use only the circular frequency and therefore from now on omit for simplicity the adjective *circular*.

impedances Z_1, Z_2, ..., Z_n in parallel: $\quad 1/Z = 1/Z_1 + 1/Z_2 + \ldots + 1/Z_n$

Examples.
14.) Inductance L and resistance R connected in series: $Z = R + j\omega L$, hence $|Z| = \sqrt{R^2 + (\omega L)^2}$ and $\arg Z = \arctan \dfrac{\omega L}{R}$
15.) Inductance L and resistance R connected in parallel: $\dfrac{1}{Z} = \dfrac{1}{R} + \dfrac{1}{j\omega L}$, hence $|Z| = \dfrac{\omega RL}{\sqrt{R^2 + (\omega L)^2}}$ $= \dfrac{\omega RL}{|R + j\omega L|^2}(L + jR)$ and $\arg Z = \arctan \dfrac{R}{\omega L}$
16.) Inductance L, capacitance C and resistance R connected in series: $Z = R + j\left(\omega L - \dfrac{1}{\omega C}\right)$, hence

$$I_0 = \frac{V_0}{|Z|} = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

Of particular interest is in the latter example the behaviour of the current I_0 for variable frequency ω at fixed values of the voltage, resistance, capacitance and inductance. Thus at $\omega = 0$ and $\omega \to \infty$ we get $I_0 = 0$, and at $\omega = 1/\sqrt{LC}$ the current has a maximum ("*current resonance*"). It is also easy to see that current and voltage are in phase at resonance.

Our simple examples only serve to illustrate the use of complex numbers in the theory of a.c. circuits. Even more important is the use of complex numbers in more advanced applications, such as the analysis of four-terminal networks, coupled circuits, transformers, filters, attenuators, delay lines, transmission lines and others.

Exercise 2. Consider an a.c. circuit with $L = 20$ mH, $R = 20\ \Omega$ and $C = 300$ pF which draws a resonance current of $I_r = 50$ mA.
(i) Find the corresponding voltage
(ii) By changing the frequency ω the current is reduced to $I = 30$ mA. Find the new frequency and the phase difference between current and voltage.
Answer: 1 V; 206 kHz; 53°.

Harmonic oscillations.

Of great importance in many fields of physics are harmonic oscillations. Examples are small oscillations of mechanical systems, electric circuits, but also oscillations of atoms and molecules, which need quantum mechanics for their theoretical treatment.

The basic form of the differential equation of the harmonic oscillator is

$$\ddot{x} + \omega_0^2 x = 0 \tag{20}$$

where $x = x(t)$ is a time-dependent displacement, $\dot{x}$ its first derivative and $\ddot{x}$ its second derivative w.r.t. time and ω_0 is the *natural frequency* of the harmonic oscillator. The general solution of this differential equation can be written in the form of

$$x = A\cos(\omega_0 t + \alpha) \tag{21}$$

where A is the *amplitude* and α the *initial phase* of the oscillation. A and α are not defined by the differential equation itself: they are integration constants which must be determined from the *initial conditions*. The initial conditions are of the form

$$x(t_0) = x_0, \qquad \text{and} \qquad \dot{x}(t = t_0) = v_0 \tag{22}$$

where x_0 and v_0 are given constants.

Exercise 3.
Show by substitution that Eq. (21) is a solution of Eq. (20) for any value of A and α.
Exercise 4.
Find the integration constants A and α of Eq. (21) if the oscillator at time $t = 0$ is at rest with a displacement x_0 from the equilibrium position.
[Answer: $A = x_0$, $\alpha = 0$]
Exercise 5.
Find the integration constants A and α of Eq. (21) if the oscillator at time $t = 0$ is in its equilibrium position and has a velocity $v_0 > 0$. (*Hint: recall that* $A > 0$)
[Answer: $A = v_0/\omega$, $\alpha = -\pi/2$]

If an element of *damping* is introduced in the system (friction in the case of mechanical systems or Ohmic resistance in a.c. circuits), then a term proportional to $\dot{x}$ must be included in the left-hand side, and if there is a sinusoidal driving force, then the right-hand side must be replaced by a *forcing term* of the form of $f \cos \Omega t$. Thus we get the differential equation of the damped, forced harmonic oscillator:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f \cos \Omega t \tag{23}$$

Solving this equation is particularly simple if one uses complex variables. Thus we write a second equation with the same left-hand side, but with $y(t)$ instead of $x(t)$, and with $f \sin \Omega t$ on the right-hand side:

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f \sin \Omega t$$

If we multiply the latter equation by $j = \sqrt{-1}$ and add it to the former equation, then we get

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f e^{j\Omega t} \tag{24}$$

where $z = x + jy$. Once a solution is found for z we can get the solution x of the original equation by simply taking the real part of z.

The particular solution of the differential equation is found by substituting into the equation $z = A \exp j\Omega t$ with unknown complex amplitude $A = |A| \exp j\alpha$, hence, after dropping the overall common factor $\exp j\Omega t$, we get

$$A(-\Omega^2 + 2j\beta\Omega + \omega_o^2) = f$$

and hence

$$|A| = \frac{f}{\sqrt{(\omega_0^2 - \Omega^2)^2 + (2\beta\Omega)^2}}, \qquad \alpha = \arctan \frac{2\beta\Omega}{\Omega^2 - \omega_0^2}$$

Finally we get the real displacement x:

$$x = \operatorname{Re} z = |A| \cos(\Omega t + \alpha)$$

The behaviour of the modulus of the amplitude as a function of the driving frequency is somewhat different from the behaviour of the current in an a.c. circuit found previously.

Indeed, for $\Omega = 0$ the amplitude does not vanish in the present case but takes on the value $|A|_{|\Omega=0} = \dfrac{f}{\omega_0}$, and the maximum occurs at $\Omega_r = \sqrt{\omega_0^2 - \beta^2}$, i.e. its position depends on the damping constant β. The reason for this different behaviour is that the correct second order differential equation of the series a.c. circuit must be written down for the *charge*, whereas the current obeys a different equation as can be seen from the relationship between charge Q and current I, viz. $I = \dfrac{dQ}{dt}$.

Exercise 6. plot a resonance curve and the curve of α vs. Ω.

General solution of the damped, forced harmonic oscillator equation

The solution $z = A\exp(j\Omega t)$ found above has no integration constants. Such a solution is called *particular solution* and we denote it from now on by z_p.

To get the *general solution* of the differential equation (24) we must add to z_p the *complementary function* z_c, i.e the general solution of the *homogeneous* equation

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = 0 \tag{25}$$

Indeed, if z_c is the complementary function of Eq. (25) and z_p is a particular solution of Eq. (24), and if $z = z_c + z_p$, then

$$\ddot{z}+2\beta\dot{z}+\omega_0 z = \frac{d^2}{dt^2}(z_c+z_p)+2\beta\frac{d}{dt}(z_c+z_p)+\omega_0^2(z_c+z_p) = \left[\ddot{z}_c + 2\beta\dot{z}_c + \omega_0^2 z_c\right]+\left[\ddot{z}_p + 2\beta\dot{z}_p + \omega_0^2 z_p\right]$$

and we recognize that the first bracket on the right-hand side vanishes by definition of z_c and the second bracket gives $f\exp(j\Omega t)$, and hence z is indeed a solution of Eq. (24). Moreover, since by assumption z_c is the general solution of Eq. (25) it must have two integration constants, and hence also $z = z_c + z_p$ has two integration constants. It is therefore the general solution of Eq. (24).

Note that the homogeneous equation describes the oscillation without forcing. Thus the complementary function describes the *free* oscillation.

It remains to show how to find the complementary function. Now, since we shall be interested in the end in the real displacement x, there is no advantage in using the equation in its complex form (25) but rather we solve the homogeneous equation corresponding to Eq. (23),

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{26}$$

To do this we substitute $x_c = \exp(\lambda t)$ into Eq. (26), hence

$$\left(\lambda^2 + 2\beta\lambda + \omega_0^2\right) e^{\lambda t} = 0$$

and since $\exp(\lambda t) \neq 0$ for any t, we can drop the exponential factor and get the quadratic equation

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0$$

(*auxiliary equation*) whose roots are

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Now we must distinguish three cases:

(i) *Light damping:* $\beta^2 - \omega_0^2 < 0$

(ii) *Heavy* or *overcritical damping:* $\beta^2 - \omega_0^2 > 0$

(iii) *Critical damping:* $\beta^2 - \omega_0^2 = 0$

and we consider each of the three cases in turn.

(i) $\beta^2 - \omega_0^2 < 0$.
Let us define the positive frequency $\omega = \sqrt{\omega_0^2 - \beta^2}$. Then the two roots can be written as

$$\lambda_{1,2} = -\beta \pm i\omega$$

and hence we get the following two solutions of Eq. (26):

$$x_1 = e^{(-\beta+i\omega)t} \qquad \text{and} \qquad x_2 = e^{(-\beta-i\omega)t}$$

The complementary function is the linear superposition of x_1 and x_2 with arbitrary integration constants c_1 and c_2, i.e.

$$x_c = e^{-\beta t}\left(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right)$$

For this to be a real quantity we can choose $c_1 = \frac{1}{2}Ae^{i\alpha}$ and $c_2 = c_1^*$, hence

$$x_c = Ae^{-\beta t}\cos(\omega t + \alpha)$$

and we have, as required, the two real arbitrary integration constants A and α to be determined from the initial conditions.

Now since a resistance in the oscillator implies that $\beta > 0$ we see that the exponential factor $\exp(-\beta t)$ causes the free oscillation x_c to die out. After a time $t = 1/\beta$ the amplitude of the free oscillation has decreased to $1/e$ of its initial value.

(ii) Next consider overcritical damping, i.e. $\beta^2 - \omega_0^2 > 0$. In this case the auxiliary equation has two distinct real roots and correspondingly we get the two solutions of Eq. (26) in the form of

$$x_1 = e^{(-\beta+\gamma)t} \qquad \text{and} \qquad x_2 = e^{-(\beta+\gamma)t}$$

where we have put $\gamma = \sqrt{\beta^2 - \omega_0^2}$. Now, since $\gamma < \beta$ both of these solutions die out exponentially, but at a different rate: x_2 dies out faster than x_1. The complementary function is given by

$$x_c = e^{-\beta t}\left(c_1 e^{\gamma t} + c_2 e^{-\gamma t}\right)$$

with real integration constants c_1 and c_2.

(iii) Finally consider critical damping, $\beta^2 - \omega_0^2 = 0$. In this case the auxiliary equation has one real double root and hence we get only one solution of the differential equation. The second, linearly independent solution of the differential equation can be shown by substitution to be of the form $x_2 = t\exp(-\beta t)$, and hence we get the complementary function in the form of

$$x_c = (c_1 + c_2 t)e^{-\beta t}$$

Complex variables

So far we have considered complex quantities essentially as constants. However already in our applications we had indications that complex quantities are at times also variable. Thus the time dependence of an oscillation was written as $\exp i\omega t = \cos(\omega t) + i\sin(\omega t)$, i.e. both the real and the imaginary parts are functions of time. This example alone shows that complex variables arise naturally in physical problems.

To indicate that a complex quantity is meant to be a variable, we shall use notation like $z = x + iy$ or $w = u + iv$ where x, y, u and v are real variables and hence z and w are complex variables. In polar form we shall use notation like $z = r\exp i\theta$ or $w = \rho\exp i\phi$.

As always in mathematics it is convenient to visualize algebraic relations by geometrical constructions. We have already seen how to represent complex numbers in the Argand diagram. Here we take this theme a step further applying it to complex variables.

Consider a complex variable $z = x+iy$. It is assumed that x and y are independent variables, i.e. that they take on independently all real values from $-\infty$ to $+\infty$. As a result the complex variable z fills the entire complex plane (z plane).

Now let us impose some condition on z. For instance, assume that we want to know all points in the complex z plane that satisfy the relation

$$|z| = 1$$

Now, we know that $|z| = \sqrt{x^2 + y^2}$, and therefore our condition on z translates into the following relation between x and y:

$$\sqrt{x^2 + y^2} = 1$$

which you recognize as the equation of the unit circle. We could have proceeded also using polar notation $z = r\exp i\theta$, in which case we would have got $|z| = r = 1$, independent of θ, which is the equation of the unit circle in polar form.

If we had the condition $|z| =$ const. then, instead of the unit circle, we would have got the equation of a family of concentric circles, one for each value of const.

A curve in the complex z plane that is obtained by imposing a condition, such as the condition $|z| = 1$, is called a *locus* of points, or just locus. Thus we can summarize the above result by saying that the locus of points $|z| =$ const. is the family of concentric circles about the origin.

Another interesting case is the locus of constant argument of z, $\arg z =$ const. It is not difficult to see that this locus is a straight line through the origin with slope const.

Example.

17.) Find the locus of points $|(z-z_0)/(z+z_0)|^2 = 1$, where z_0 is a complex constant, $z_0 = x_0+iy_0$.
Answer: straight line through the origin, $xx_0 + yy_0 = 0$.