Selected Topics in Physics

a lecture course for 1st year students

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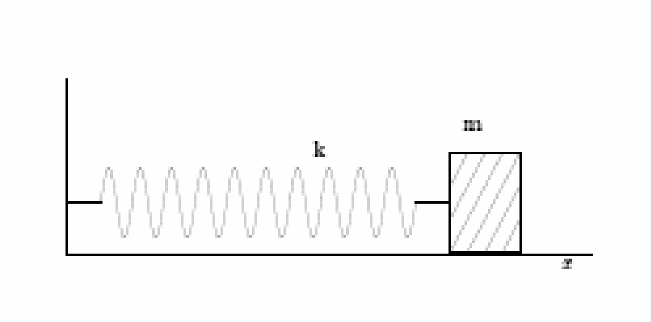
Lecture 2: Oscillations

- simple harmonic oscillations;
- coupled oscillations; beats;
- damped oscillations;
- forced oscillations.
- pendulum

2 Oscillations

2.1 Simple Harmonic Oscillations

A mass m is attached to a spring as shown in the figure, and slides without friction on a horizontal surface. Its movement is confined to one dimension which we choose to be the x axis.



The force exerted on the mass is

$$F = -kx$$

where k is the ***spring constant***.

The equation of motion is

$$m\ddot{x} = -kx$$

or with $\omega^2 = k/m$

$$\ddot{x} + \omega^2 x = 0$$

We solve this differential equation by putting $x = e^{\lambda t}$

hence

$$\left(\lambda^2 + \omega^2\right)e^{\lambda t} = 0$$

and since $e^{\lambda t} \neq 0$ we get $\lambda = \pm i\omega$ *i.e.* we have two solutions.

The general solution of the DEq is a linear superposition of these solutions:

$$x = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$

and with Euler's formula $e^{\pm i\alpha} = \cos\alpha \pm i\sin\alpha$

we get

$$x = (c_1 + c_2)\cos\omega t + i(c_1 - c_2)\sin\omega t$$

and with

$$c_1 = \frac{1}{2}(a - ib); \quad c_2 = \frac{1}{2}(a + ib)$$

we have finally

$$x = a\cos\omega t + b\sin\omega t$$

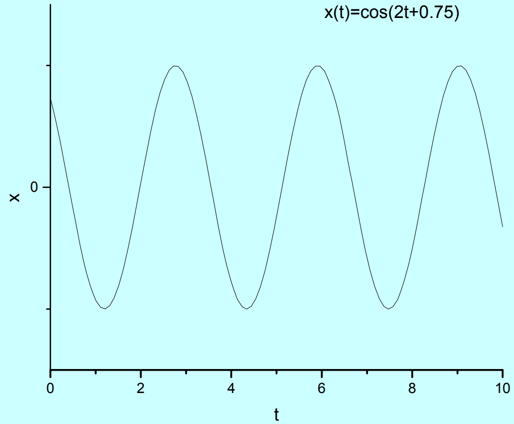
where a and b are two integration constants to be defined by the initial conditions.

An alternative form of the solution is

$$x = A\cos(\omega t + \alpha)$$

A is the ***amplitude*** and α is the ***initial phase.***

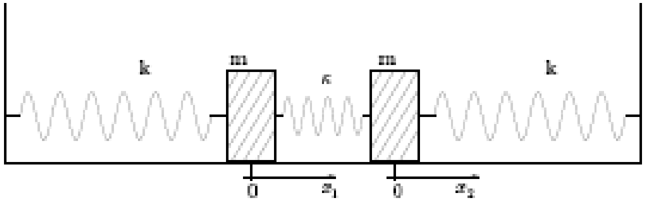
The expression $\omega t + \alpha$ is the (instantaneous) ***phase*** of oscillation.



Shown in the figure is the s.h.o. $x = A\cos(2t + 0.75)$

2.2 Coupled Oscillations

Consider two masses, each connected to a wall through a spring of spring constant k, coupled to each other by a spring of spring constant κ and sliding without friction on a horizontal surface.



The displacements of the masses from their equilibrium positions are x_1 and x_2, respectively.

The equations of motion are

$$m\ddot{x}_1 = -kx_1 - \kappa\left(x_1 - x_2\right) \qquad (2.1)$$

$$m\ddot{x}_2 = -kx_2 + \kappa\left(x_1 - x_2\right) \qquad (2.2)$$

To solve these simultaneous DEqs we take their sum and difference:

(2.1)+(2.2): $$m\left(\ddot{x}_1+\ddot{x}_2\right)=-k\left(x_1+x_2\right) \qquad (2.3)$$

(2.1)-(2.2): $$m\left(\ddot{x}_1-\ddot{x}_2\right)=-\left(k+2\kappa\right)\left(x_1-x_2\right) \qquad (2.4)$$

and then define two new variables:

$$y=x_1+x_2 \quad \text{and} \quad z=x_1-x_2 \qquad (2.5)$$

hence

$$m\ddot{y}=-ky \qquad (2.6)$$

$$m\ddot{z}=-\left(k+2\kappa\right)z \qquad (2.7)$$

i.e. the equations (2.1) and (2.2) are ***decoupled*** into two s.h.o. equations.

Define:

$$\omega^2=k/m \quad \text{and} \quad \omega_1^2=\left(k+2\kappa\right)/m \qquad (2.8)$$

then our eqs (2.6) and (2.7) become

$$\ddot{y} + \omega^2 y = 0$$

$$\ddot{z} + \omega_1^2 z = 0$$

whose solutions are

$$y = A\cos(\omega t + \alpha)$$

$$z = B\cos(\omega_1 t + \beta)$$

and going back to the coordinates x_1 and x_2 by Eq. (2.5), we get

$$x_1 = \frac{1}{2}\left[A\cos(\omega t + \alpha) + B\cos(\omega_1 t + \beta)\right] \qquad (2.9)$$

$$x_2 = \frac{1}{2}\left[A\cos(\omega t + \alpha) - B\cos(\omega_1 t + \beta)\right] \qquad (2.10)$$

This completes the derivation of the general solution of the equations of motion.

To discuss particular cases we must specify the initial conditions.

For instance, let us assume that initially, *i.e.* at time t=0, both masses are at rest, with the right-hand mass in its equilibrium position and the left-hand mass displaced to x_1=a:

$$\text{IC:}\quad t=0,\quad x_1=a,\, x_2=0,\, \dot{x}_1=\dot{x}_2=0$$

Substituting into (2.9) and (2.10) we get

$$A\cos\alpha + B\cos\beta = 2a \quad (2.11)$$

$$A\cos\alpha - B\cos\beta = 0 \quad (2.12)$$

and

$$A\omega\sin\alpha + B\omega_1\sin\beta = 0 \quad (2.13)$$

$$A\omega\sin\alpha - B\omega_1\sin\beta = 0 \quad (2.14)$$

which we can satisfy with $\alpha=\beta=0$ (but not with $A=0$ or $B=0$: Exercise!)

and hence $B=A$, thus

$$x_1 = \frac{1}{2}A\left[\cos(\omega t) + \cos(\omega_1 t)\right]$$

$$x_2 = \frac{1}{2}A\left[\cos(\omega t) - \cos(\omega_1 t)\right]$$

or, using a well known trigonometric identity, we get

$$x_1 = A\cos\left(\frac{\omega+\omega_1}{2}t\right)\cos\left(\frac{\omega_1-\omega}{2}t\right) \qquad (2.15)$$

$$x_2 = A\sin\left(\frac{\omega+\omega_1}{2}t\right)\sin\left(\frac{\omega_1-\omega}{2}t\right) \qquad (2.16)$$

An interesting case arises when the coupling is weak, *i.e.* when the spring constant κ is much less than k.

Recall: $\omega = \sqrt{k/m}$ and $\omega_1 = \sqrt{(k+2\kappa)/m}$

hence

$$\omega + \omega_1 \simeq 2\omega \quad \text{and} \quad \omega_1 - \omega \simeq \omega \frac{\kappa}{k} \ll \omega$$

and hence Eqs. (2.15) and (2.16) become

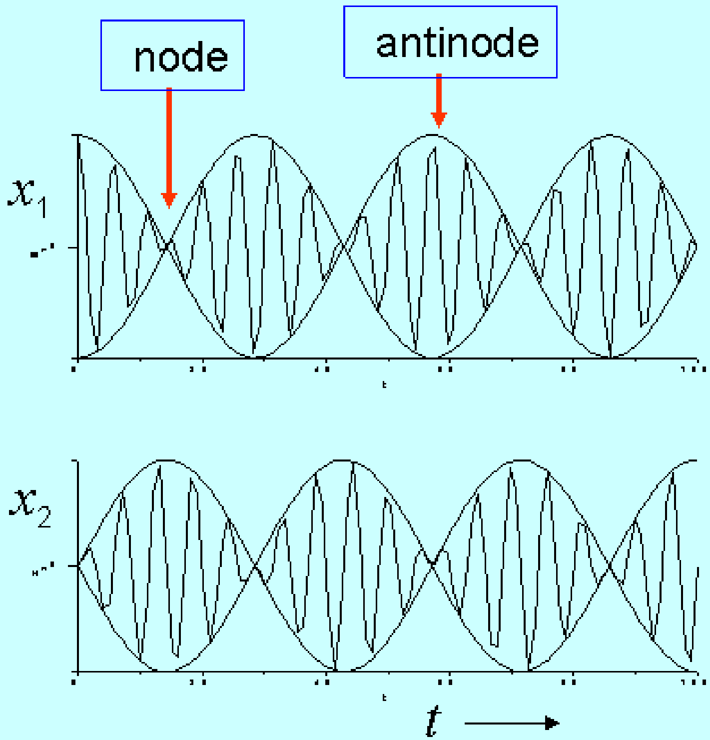
$$x_1 = A\cos(\delta t)\cos(\omega t) \tag{2.17}$$

$$x_2 = A\sin(\delta t)\sin(\omega t) \tag{2.18}$$

Here the rapid oscillation with circular frequency ω is ***modulated*** by the slow oscillation of frequency $\delta = \omega\kappa/2k$.

It is usual to consider the expression $A\cos(\delta t)$ to be a slowly varying amplitude.

The functions (2.17) and (2.18) are shown in the next figure.



The initial energy of oscillator 1, which was residing in the springs, is transferred to oscillator 2. At the first node of oscillator 1 all energy has been transferred to oscillator 2, then the process is reversed.

This phenomenon is called ***beats***.

2.3 Damped Oscillations

Realistically, there is always some amount of damping resistance present in every oscillation. In a system like the one of 2.1, the resistance is the friction between the mass and the supporting surface.

Intuitively we say that the resistance depends on the speed of the mass but not on its position. At zero velocity the resistance must vanish, therefore it is reasonable to set

$$R\left(\dot{x}\right) = R'\left(0\right)\dot{x} + O\left(\dot{x}^2\right)$$

and for this to be a resistance we must have $R'\left(0\right) < 0$

hence, neglecting the terms of order x^2, we get the equation of motion of the damped harmonic oscillator in the form of

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0} \qquad (2.17)$$

with

$$2\beta = -R'\left(0\right)/m \quad \text{and} \quad \omega_0^2 = k/m$$

Equation (2.17) is a ***linear differential equation*** with ***constant coefficients***.

To solve it, we make the substitution (***Ansatz***): $x(t) = e^{\lambda t}$

hence

$$\left(\lambda^2 + 2\beta\lambda + \omega_0^2\right)e^{\lambda t} = 0$$

but $e^{\lambda t} \neq 0$

and therefore we get the ***characteristic equation***

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0 \qquad (2.18)$$

whose two roots are

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \qquad (2.19)$$

The general solution of DEq. (2.17) is therefore

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \qquad (2.20)$$

Now we must distinguish three cases:

$$(i) \quad \text{undercritical damping:} \quad \beta^2 - \omega_0^2 < 0$$

$$(ii) \quad \text{critical damping:} \quad \beta^2 - \omega_0^2 = 0$$

$$(iii) \quad \text{overcritical damping:} \quad \beta^2 - \omega_0^2 > 0$$

(i) define: $\omega^2 = \omega_0^2 - \beta^2$ hence $\lambda_{1,2} = -\beta \pm i\omega$

$$x(t) = c_1 e^{(-\beta + i\omega)t} + c_2 e^{(-\beta - i\omega)t} = e^{-\beta t}\left(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right)$$

Let $c = c_1$ and $c_2 = c^*$ $\left(c^* \text{ is the complex conjugate of } c\right)$

hence

$$x(t) = e^{-\beta t}\left(c e^{i\omega t} + c^* e^{-i\omega t}\right) = 2e^{-\beta t}\,\mathrm{Re}\left(c e^{i\omega t}\right)$$

or putting

$$c = \frac{1}{2} A e^{i\alpha} \quad \text{with real} \quad A \text{ and } \alpha$$

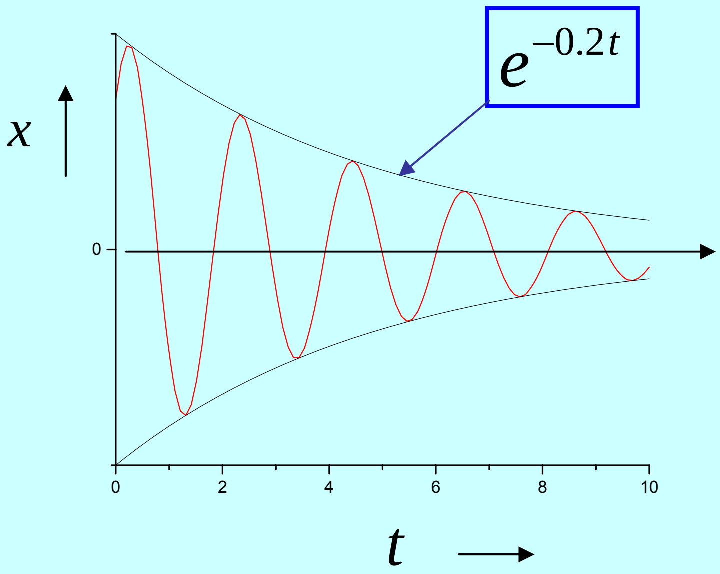
we get

$$\boxed{x(t) = A e^{-\beta t} \cos(\omega t + \alpha)}$$

where A and α are integration constants.

Example of undercritically damped oscillation:

$$A = 1, \quad \omega = 3, \quad \beta = 0.2, \quad \alpha = -0.8$$



(ii) critical damping: $\beta^2 - \omega_0^2 = 0$

the problem here is that we do not have two linearly independent solutions:

$$x_2 = x_1$$

So leave x_1 at the value that comes out of the calculation:

$$x_1 = e^{-\beta t}$$

But to construct the second, linearly independent solution, use the following:

define: $\Lambda = \beta^2 - \omega_0^2 > 0 \quad \left(\Lambda \to 0 \quad \text{at the end}\right)$

and put

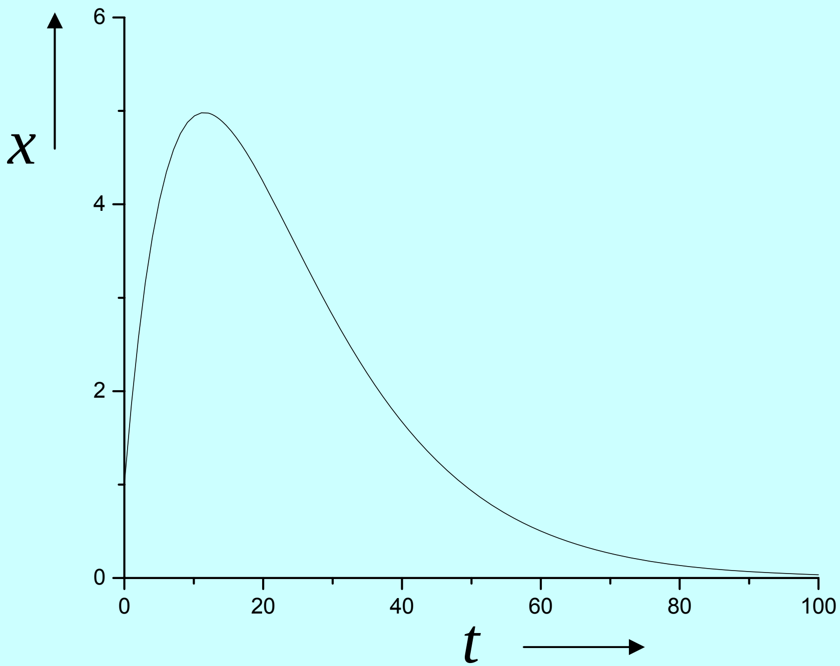
$$x_2 = e^{-\beta t} \lim_{\Lambda \to 0} \frac{1}{2\Lambda}\left(e^{\Lambda t} - e^{-\Lambda t}\right) = t e^{-\beta t}$$

and then the general solution takes on the form

$$\boxed{x(t) = \left(c_1 + c_2 t\right) e^{-\beta t}}$$

Example of critically damped system:

$$x = (1+t)e^{-0.08t}$$



Critical damping is of interest if one wants to make the system settle down as fast as possible after a disturbance, such as in the wheel suspensions of vehicles and in many measuring instruments, *e.g.* in ballistic galvanometers.

(iii) overcritical damping: $\beta^2 - \omega_0^2 > 0$

define: $\Lambda^2 = \beta^2 - \omega_0^2 > 0 \quad (\Lambda < \beta!)$

hence

$$x_1 = e^{(-\beta+\Lambda)\,t}; \qquad x_2 = e^{(-\beta-\Lambda)\,t}$$

and hence the general solution:

$$x(t) = e^{-\beta t}\left(c_1 e^{\Lambda t} + c_2 e^{-\Lambda t}\right)$$

and for sufficiently large times t we get an exponentially decaying displacement:

$$\boxed{x(t)_{|\Lambda t \gg 1} \sim c_1 e^{-(\beta-\Lambda)t}}$$

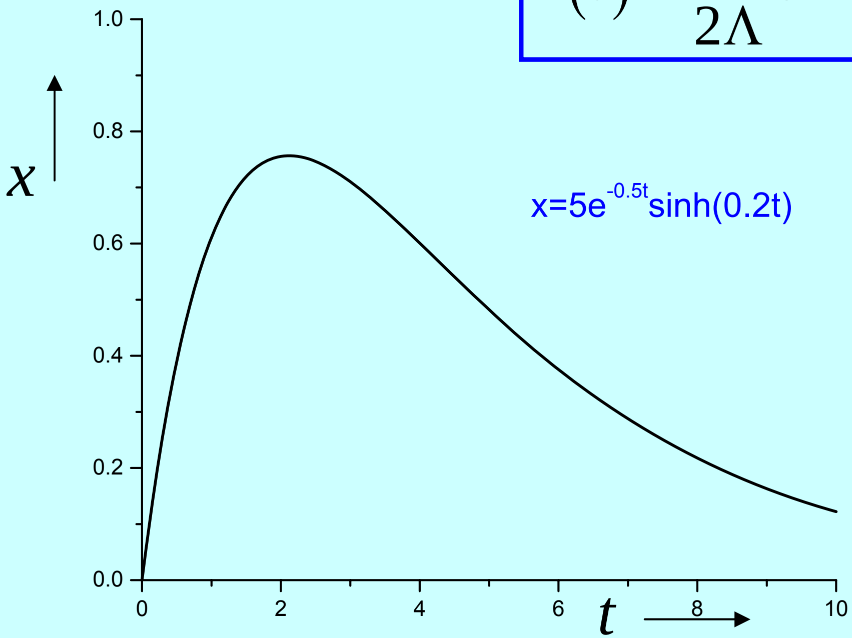
Example of overcritically damped system:

IC: at $t = 0: \ x = 0$ and $\dot{x} = v > 0$

hence $c_1 = \frac{v}{2\Lambda}, \quad c_2 = -c_1$

thus

$$x(t) = \frac{v}{2\Lambda} e^{-\beta t} \left(e^{\Lambda t} - e^{-\Lambda t} \right) = \frac{v}{\Lambda} e^{-\beta t} \sinh \Lambda t$$



2.4 Forced Oscillations

Oscillating system with sinusoidal forcing term:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f\cos\Omega t \qquad (2.21)$$

It is convenient to write down a second, similar equation:

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f\sin\Omega t \qquad (2.22)$$

If we multiply Eq. (2.22) by i, then add it to Eq. (2.21), define the new variable

$$z = x + iy$$

and use Euler's formula $\cos\Omega t + i\sin\Omega t = e^{i\Omega t}$

then we get the following DEq.:

$$\boxed{\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = fe^{i\Omega t}} \qquad (2.23)$$

To solve this DEq we make the Ansatz: $z = Ae^{i\Omega t}$

hence

$$A\left(-\Omega^2 + 2i\beta\Omega + \omega_0^2\right)e^{i\Omega t} = fe^{i\Omega t}$$

and after cancellation of the exponential factors we get

$$A = \frac{f}{\omega_0^2 - \Omega^2 + 2i\beta\Omega}$$

and with $A = |A| e^{i\alpha}$

we get the ***modulus of the amplitude*** and the ***phase***:

$$|A| = \frac{f}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\beta^2\Omega^2}} \qquad (2.24)$$

$$\tan\alpha = \frac{\operatorname{Im} A}{\operatorname{Re} A} = \frac{2\beta\Omega}{\Omega^2 - \omega_0^2} \qquad (2.25)$$

The general solution of DEq (2.23) is

$$z = Ae^{i\Omega t} + e^{-\beta t}\left(a\cos\omega t + b\sin\omega t\right)$$

where $z_t = e^{-\beta t}\left(a\cos\omega t + b\sin\omega t\right)$ is the ***transient oscillation***

For $\beta t \gg 1$ we get the ***steady state solution***:

$$z = Ae^{i\Omega t} = |A|e^{i(\Omega t+\alpha)}$$

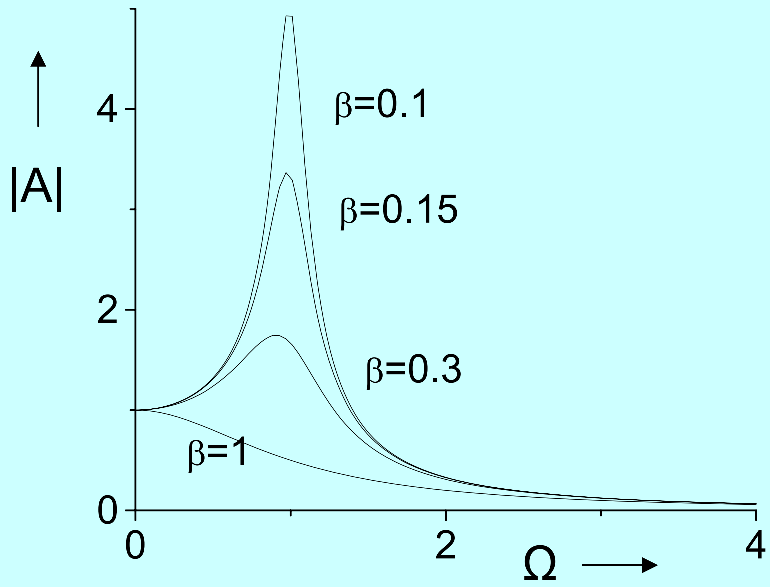
and hence the steady state solution of the original DEq (2.21):

$$\boxed{x = \operatorname{Re} z = |A|\cos\left(\Omega t + \alpha\right)}$$

where the amplitude and phase are given by Eqs (2.24) and (2.25).

The amplitude is shown in the next figure as function of the forcing frequency Ω.

Amplitude of forced oscillation as function of the forcing frequency Ω



We see that the amplitude has a peak which gets more pronounced at smaller resistance. Such a behaviour is called a ***resonance***.

The position of the maximum of |A| is the ***resonance frequency;*** it is given by

$$\Omega_R = \sqrt{\omega_0^2 - 2\beta^2} \qquad (2.26)$$

and the height of the resonance peak is given by

$$\left|A\right|_R = \frac{f}{2\beta\sqrt{\omega_0^2 - \beta^2}} \qquad (2.27)$$

By Eq. (2.26) reality of the resonance frequency implies

$$\beta \le \omega_0 / \sqrt{2}$$

and therefore also $|A|_R$ is seen to be real for all allowed values of β.

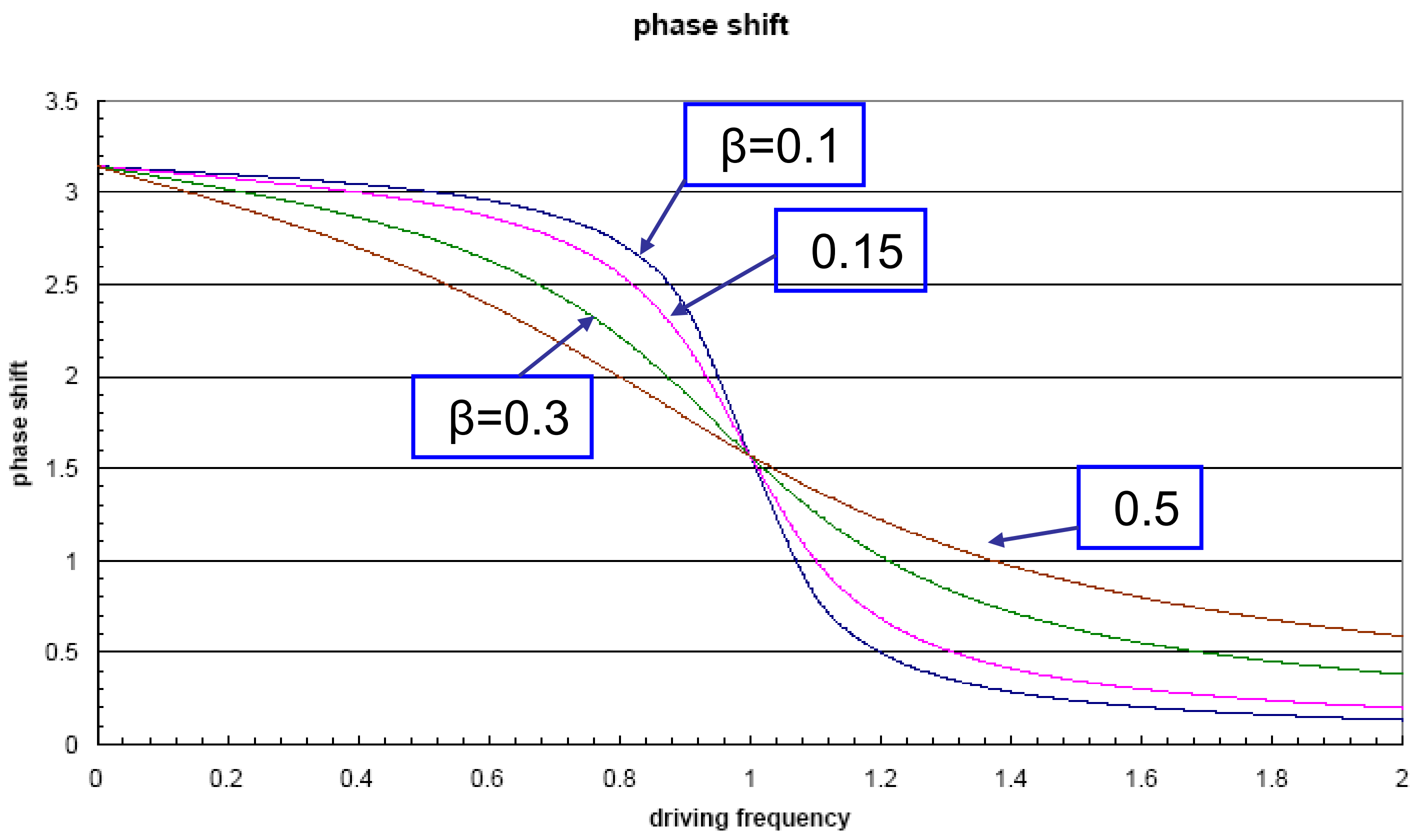
There is no resonance if $\beta \ge \omega_0/\sqrt{2}$

For $\beta \rightarrow 0$, *i.e*. in the absence of damping, the amplitude tends to infinity as we approach resonance. In practice the system breaks up before resonance is reached.

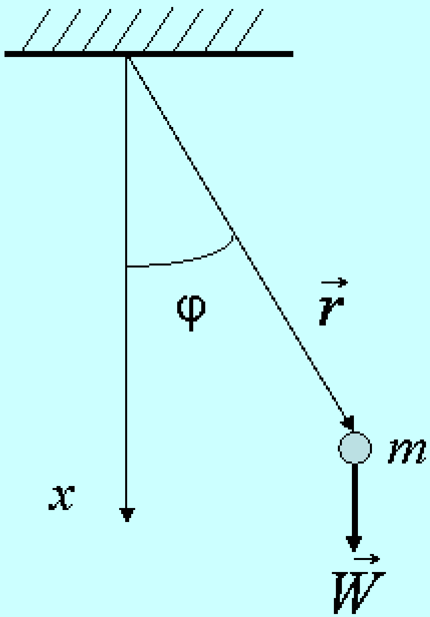
Of interest is also the behaviour of the phase shift α as a function of the driving frequency.

Phase shift α as function of the driving frequency Ω for $\omega_0=1$

Note that α is passing through $\pi/2$ at $\Omega=\omega_0$



2.5 Pendulum



A pendulum consists of a mass m attached to one end of a weightless rigid rod which is suspended without friction from a rigid support

Two forces act on m: its weight and the string tension (not shown in the drawing), which is equal in magnitude and opposite in direction to the component of the weight in the direction of the string; therefore we have

$$m\ddot{\vec{r}} = \vec{W} + \vec{T} \qquad (2.28)$$

with

$$\vec{T} = -\left(\vec{W} \cdot \hat{r}\right)\hat{r}$$

and $\vec{W} = mg\hat{x}$ where $\hat{x}$ is the unit vector in x direction

Now take the cross product of (2.28) with the vector $\vec{r}$

On the left-hand side we have

$$\vec{r}\times\ddot{\vec{r}}=\frac{d}{dt}\left(\vec{r}\times\dot{\vec{r}}\right)=l^{2}\ddot{\varphi}\,\hat{n}$$

and on the right-hand side we have

$$\vec{r}\times\left(\vec{W}+\vec{T}\right)=\vec{r}\times\vec{W}=-mgl\sin\varphi\,\hat{n}$$

since the vectors $\vec{r}$ and $\vec{T}$ are collinear and therefore their cross product is equal to nought.

l is the length of the pendulum rod and

$\hat{n}$ is a unit vector perpendicular to $\vec{r}$ and $\vec{W}$

Thus the equation of motion takes on the form of

$$\ddot{\varphi}=-\left(g/l\right)\sin\varphi$$

or with $\omega^{2}=g/l$

$$\boxed{\ddot{\varphi}+\omega^{2}\sin\varphi=0} \qquad (2.29)$$

Equation (2.29) is a nonlinear second order differential equation. The standard way of solving it is slightly artificial: we begin by multiplying the equation by $\dot{\varphi}$

$$\dot{\varphi}\ddot{\varphi} + \omega^2 \dot{\varphi} \sin\varphi = 0$$

and we note that $\dot{\varphi}\ddot{\varphi} = \frac{d}{dt}\left(\frac{\dot{\varphi}^2}{2}\right)$ and $\dot{\varphi}\sin\varphi = -\frac{d}{dt}\cos\varphi$

hence

$$\frac{d}{dt}\left(\frac{\dot{\varphi}^2}{2} - \omega^2 \cos\varphi\right) = 0$$

and hence

$$\frac{\dot{\varphi}^2}{2} - \omega^2 \cos\varphi = C$$

where C is the integration constant. We can fix C by putting φ to the maximum angle φ_0, which we call the ***amplitude***. For $\varphi = \varphi_0$ the angular velocity is zero, and hence

$$C = -\omega^2 \cos\varphi_0$$

and hence we get

$$\frac{\dot{\varphi}^2}{2} = \omega^2 \left(\cos\varphi - \cos\varphi_0\right)$$

or

$$\dot{\varphi} = \pm\sqrt{2\omega^2 \left(\cos\varphi - \cos\varphi_0\right)}$$

and hence

$$dt = \pm\frac{d\varphi}{\sqrt{2\omega^2 \left(\cos\varphi - \cos\varphi_0\right)}}$$

Now the period of oscillation T is the time of a full swing from $-\varphi_0$ to φ_0 and back. Therefore, making also use of the symmetry of the system, we get

$$T = \frac{4}{\omega}\int_0^{\varphi_0} \frac{d\varphi}{\sqrt{2\left(\cos\varphi - \cos\varphi_0\right)}}$$

This integral belongs to the class of elliptic integrals. They cannot be expressed in terms of elementary functions. The standard way of handling such an integral is the following.

First we apply the identity

$$\cos\varphi = 1 - 2\sin^2\frac{\varphi}{2}$$

hence

$$T = \frac{4}{\omega}\int_0^{\varphi_0}\frac{d\left(\varphi/2\right)}{\sqrt{\sin^2\left(\varphi_0/2\right) - \sin^2\left(\varphi/2\right)}}$$

and then we make the substitution

$$\sin\frac{\varphi}{2} = \sin\frac{\varphi_0}{2}\sin u$$

and we note that u=0 for φ=0 and $u=\pi/2$ for $\varphi=\varphi_0$, hence

$$T = \frac{4}{\omega}\int_0^{\pi/2}\frac{du}{\sqrt{1 - m\sin^2 u}} \qquad (2.30)$$

where $m = \sin^2\left(\varphi_0/2\right)$ is the ***parameter*** of the elliptic integral.

For amplitudes small enough for m to be negligibly small we get

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

which is the well known result that we get more easily if we approximate sin φ by φ in Eq. (2.29). But the work done to get Eq. (2.30) pays when we want to apply our result to large amplitudes. Then we can proceed as follows: we note that $m\sin^2 u < 1$ for $\varphi_0 < \pi$, and hence we expand:

$$\frac{1}{\sqrt{1 - m\sin^2 u}} = 1 + \frac{1}{2} m\sin^2 u + \frac{1\cdot 3}{2\cdot 4}\left(m\sin^2 u\right)^2 + \ldots$$

and we get a first correction if we neglect terms of order m^2:

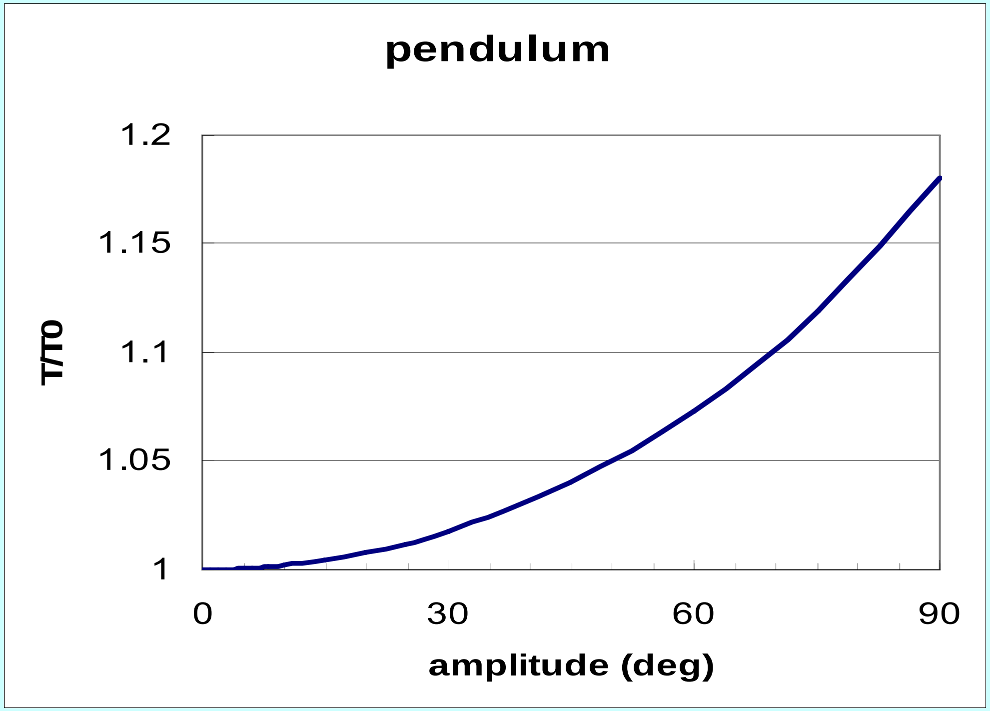
$$T = T_0\left(1 + \frac{1}{4}\sin^2\frac{\varphi_0}{2}\right)$$

with $T_0 = 2\pi\sqrt{l/g}$

For an amplitude of φ_0 = 6 degrees we have

$$\frac{1}{4}\sin^2\frac{\varphi_0}{2} \cong 0.0007$$

and we see that neglecting this correction is justified. But for large amplitudes the power series is inconvenient since one needs to carry a large number of terms. Then it is simpler to do the integration numerically to any required accuracy. The results are shown in the next figure:



In the figure the ratio $T\,/\,T_0$ is shown as a function of the amplitude φ_0. One can see that even at an amplitude of 90 degrees the period of oscillation is increased by less than 20%.

Problems for Lecture 1:

1.) Show that the speed of a satellite in a circular orbit about the Earth is given by

$$v = R_E \sqrt{g/r}$$

where R_E =6378 km is the radius of the Earth (to be assumed spherical), g=*9.8 ms*$^{-2}$ is the acceleration due to gravity and r is the radius of the orbit.

Find the speed of the International Space Station, taking its altitude to be 330 km (the actual orbit of the ISS is not perfectly circular but nearly so).

2.) Find the radius of the geostationary orbit about the Earth.
(Answer: 4.22×10^7 m)