

Selected Topics in Physics

a lecture course for 1st year students

by W.B. von Schlippe

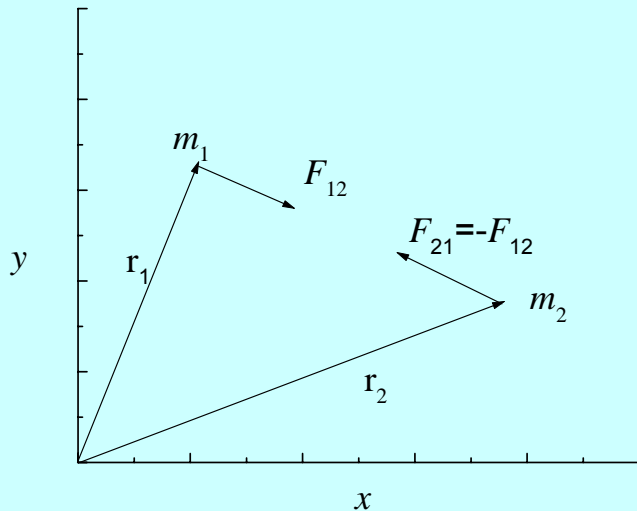
Spring Semester 2007

Lecture 4

Kepler Problem (continued)

Kepler Problem (continued)

The Kepler problem is the problem of the motion of two masses in gravitational interaction.



For two point masses, m_1 and m_2 , located at \mathbf{r}_1 and \mathbf{r}_2 , respectively the gravitational force exerted by m_2 on m_1 is

$$\vec{F}_{12} = G \frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_1 - \vec{r}_2|}$$

and the force exerted by m_1 on m_2 is by Newton's third law

$$\vec{F}_{21} = -\vec{F}_{12}$$

The equations of motion for the two bodies are:

$$m_1 \ddot{\vec{r}}_1 = G \frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_2 - \vec{r}_1) \quad (4.1)$$

$$m_2 \ddot{\vec{r}}_2 = -G \frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_2 - \vec{r}_1) \quad (4.2)$$

There follows the **conservation of total momentum**:

$$\vec{P} = M \dot{\vec{R}} = m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = \text{constant}; \quad M = m_1 + m_2$$

where $\vec{R} = (m_1 \vec{r}_1 + m_2 \vec{r}_2) / (m_1 + m_2)$ is the radius vector of the centre of mass.

Equivalent with the conservation of total momentum is the following equation of motion:

$$\dot{\vec{P}} = M \ddot{\vec{R}} = 0 \quad (4.3)$$

Then we have found a second eqn of motion in terms of the *relative coordinate* $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$\mu \ddot{\vec{r}} = -\frac{\gamma}{r^2} \hat{r} \quad \text{with} \quad \gamma = G\mu M \quad (4.4)$$

which is the equation of motion of a particle of mass μ in the gravitational field of a mass M that is at rest at the origin of the coordinate system.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{is the reduced mass.}$$

Then we have found two more conservation laws:

conservation of energy:

$$\frac{1}{2} \mu \dot{\vec{r}}^2 - \frac{\gamma}{r} = E \quad (4.5)$$

conservation of angular momentum

$$\begin{aligned} \vec{L} &= \mu \vec{r} \times \dot{\vec{r}} \\ &= \mu r^2 \dot{\phi} \hat{n} = \text{constant} \end{aligned} \quad (4.6)$$

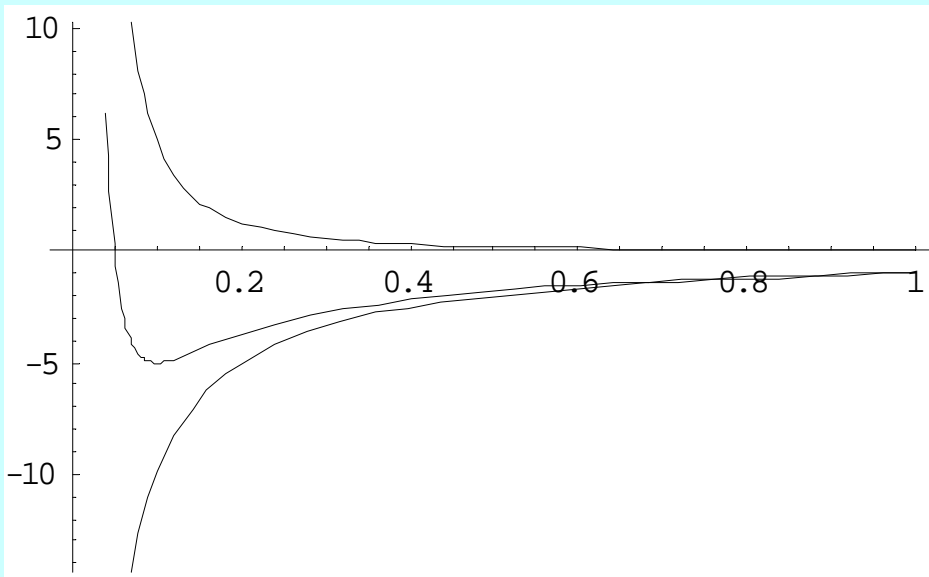
$$\hat{n} = \frac{\vec{r} \times \dot{\vec{r}}}{|\vec{r} \times \dot{\vec{r}}|}$$

hence

$$\frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{\gamma}{r} = E \quad (4.7)$$

effective P.E.:

$$V_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} - \frac{\gamma}{r}$$



which has a minimum at

$$r_0 = L^2 / \mu\gamma \quad (4.8)$$

$$V_{\text{eff}}(r_0) = -\frac{\mu\gamma^2}{2L^2} \quad (4.9)$$

hence

$$E \geq -\frac{\mu\gamma^2}{2L^2}$$

The equal sign corresponds to circular motion with radius

$$\boxed{r = -\gamma/2E} \quad (4.10)$$

For $E_{\min} < E < 0$ the motion lies between a minimum and a maximum value of r , which we get from the condition

$$T = 0 \quad \text{hence} \quad E = V_{\text{eff}}(r)$$

i.e.

$$E = \frac{L^2}{2\mu r^2} - \frac{\gamma}{r}$$

hence

$$r_{\min, \max} = -\frac{\gamma}{2E} \pm \sqrt{\left(\frac{\gamma}{2E}\right)^2 + \frac{L^2}{2\mu E}} \quad (4.11)$$

Then we eliminate the time from Eq. (4.7) using (4.6) and end up with two Deqs.:

$$\dot{\phi} = \frac{L}{\mu r^2} \quad (4.6a)$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2\mu\gamma}{L^2}u = \frac{2\mu E}{L^2}, \quad u = \frac{1}{r} \quad (4.12)$$

Eq. (4.12) has the solution

$$u = \frac{1}{r} = \frac{\mu\gamma}{L^2} + A \cos(\phi + \alpha) \quad (4.13)$$

or

$$r = \frac{L^2/\mu\gamma}{1 + A(L^2/\mu\gamma)\cos(\phi + \alpha)} \quad (4.14)$$

This is the eqn of the trajectory; the time dependence can be found by substituting r into (4.6a) and carrying out one final integration.

The constants A and α are integration constants of the 2nd order DEq. (3.28), but the solution we have found must satisfy the 1st order DEq. (3.27), which has only one integration constant.

Substituting the solution (4.13) into (4.12) we get:

$$A^2 \sin^2(\varphi + \alpha) + \left(\frac{\mu\gamma}{L^2} + A \cos(\varphi + \alpha) \right) \left(A \cos(\varphi + \alpha) - \frac{\mu\gamma}{L^2} \right) = \frac{2\mu E}{L^2}$$

hence


$$A^2 = \frac{2\mu E}{L^2} + \left(\frac{\mu\gamma}{L^2} \right)^2$$

And since

$$E \geq -\mu\gamma^2 / 2L^2$$

$$A^2 \geq 0$$

Thus there are solutions of four classes:

<p>(i) $A = 0;$ (ii) $0 < A < \gamma\mu/L^2;$ (iii) $A = \gamma\mu/L^2;$ (iv) $A > \gamma\mu/L^2.$</p>		<p>$E = E_{\min} = -\mu\gamma^2/2L^2,$ $E_{\min} < E < 0,$ $E = 0,$ $E > 0.$</p>
--	--	--

(i) $A = 0$ is the circular motion of radius

$$r = L^2 / \mu\gamma = -\gamma / 2E$$

Tracing our transformations back to \mathbf{r}_1 and \mathbf{r}_2 in the CMS, we have

$$m_2 \vec{r}_2(t) = -m_1 \vec{r}_1(t), \quad \longrightarrow \quad r_1(t) / r_2(t) = m_2 / m_1$$

Conclusion: the two bodies move in concentric circles of radii, inversely proportional to their masses, and are always in opposition.

$$(ii) \quad 0 < |A| < \gamma\mu/L^2$$

In this case r remains finite for all values of φ

and we note that

$$r(\varphi + 2\pi) = r(\varphi)$$

i.e. the orbit is closed.

From the theory of conic sections we know that it is an ellipse.

If we choose $\alpha = 0$, then the major axis of the ellipse corresponds to $\varphi = 0$, and we have

$$r_{\max} + r_{\min} = 2a = -\gamma/E$$

and hence

$$a = -\gamma/2E$$

The distance r at $\varphi=\pi/2$ is the *semi-latus rectum*:

$$\ell = \frac{L^2}{\gamma\mu} \quad (4.15)$$

and the equation of the ellipse becomes

$$r = \frac{\ell}{1 + \varepsilon \cos \varphi} \quad (4.16)$$

with the eccentricity $\varepsilon = \sqrt{1 - \ell/a}$ and $A > 0$ was assumed.

Had we chosen $A < 0$ (but of the same magnitude), then we would have got instead of (4.16) (**Exercise!**):

$$r = \ell / (1 - \varepsilon \cos \varphi)$$

which is the same ellipse but referred to the other focus.

Now recall once more the equation of angular momentum conservation:

$$L = \mu r^2 \dot{\phi}$$

If we divide by 2μ and multiply by dt , then we get

$$\frac{L}{2\mu} dt = \frac{1}{2} r^2 d\phi = dF$$

which is the area swept by the radius vector in the time interval dt , and since by angular momentum conservation $L/2\mu$ is constant, we can integrate to get the area swept by the radius vector in a finite interval of time:

$$\Delta F = (L/2\mu) \Delta t$$

and if we choose that time to be the time T of one revolution, then F is the area of the ellipse,

$$F = (L/2\mu)T = \pi ab$$

where b is the semi-minor axis: $b = \sqrt{al}$

Using (4.15) to eliminate the angular momentum L we get

$$\frac{1}{2} \sqrt{\frac{\gamma}{\mu}} T = \pi a^{3/2}$$

and with

$$\gamma = G\mu M$$

after a little calculation:

$$\boxed{\frac{a^3}{T^2} = \frac{GM}{4\pi^2}} \quad (4.17)$$

Now, by Kepler's third law, the ratio a^3/T^2 is the same for all planets known by his time. But we see that in Eq. (4.17) M is the sum of the masses of the sun and the planet. We conclude that the mass of the sun is so much greater than the masses of the planets that the discrepancy is not observable.

Therefore, applying (4.17) to the solar system, M is the mass of the sun within observational accuracy.

Then, since G is measured in terrestrial experiments (e.g. using a *torsion balance*) and a and T are found by astronomical observation, Eq. (3.32) gives us the solar mass:

$$M = 4\pi^2 \frac{a^3}{GT^2}$$

and hence with values appropriate for the earth:

$$a = 1 \text{ a.u.} = 1.4960 \times 10^{11} \text{ m}, \quad T = 1 \text{ yr} = 3.1557 \times 10^7 \text{ s},$$

$$G = 6.6726 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

$$M_{\odot} = 1.9893 \times 10^{30} \text{ kg}$$

$$(iii) \quad |A| = \gamma\mu/L^2$$

$$r = \frac{\ell}{1 + \cos \varphi}$$

This is the equation of a parabola in polar coordinates and we have

$$r \rightarrow \infty \quad \text{for} \quad \varphi \rightarrow \pm\pi$$

$$(iv) \quad |A| > \gamma\mu/L^2.$$

$$r = \frac{\ell}{1 + \varepsilon \cos \varphi}, \quad \varepsilon > 1$$

This is the equation of a hyperbola in polar coordinates.

The requirement of positivity of r implies

$$\cos \varphi > -1/\varepsilon$$

In astronomical terms hyperbolic orbits are the orbits of non-recurring comets.

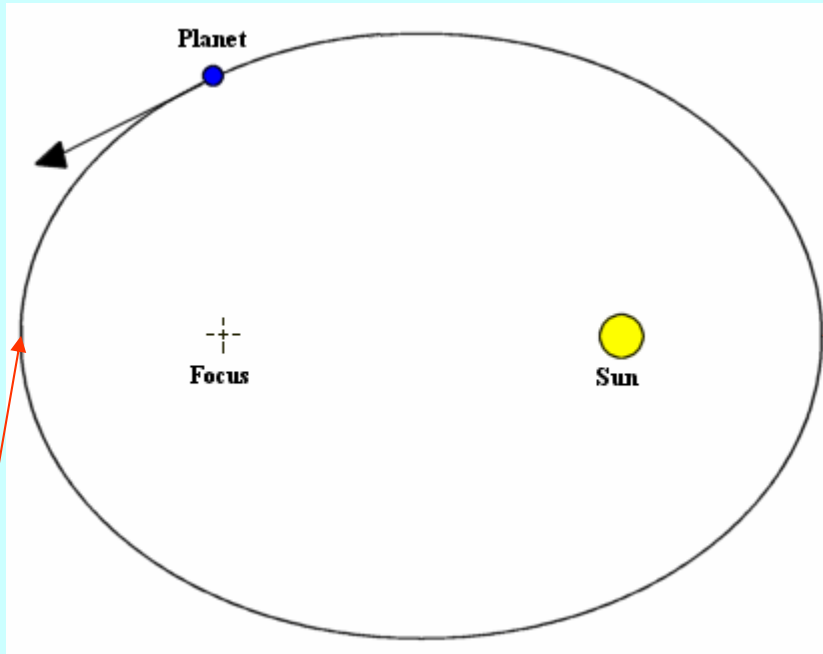
Hyperbolic orbits also arise in the case of repulsive forces, such as the Coulomb force between a gold nucleus and an alpha particle (**Rutherford experiment!**); in this case the equation of motion is

$$\mu \ddot{\vec{r}} = \frac{zZe^2}{r^2} \hat{r}$$

where ze is the charge of the projectile and Ze the charge of the target, which is supposed to be stationary; e is the elementary charge (= charge of a proton).

In the case of Rutherford scattering we have $z = 2$ and $Z = 79$.

Of interest is to find the **distance of closest approach**, *i.e.* the shortest distance between the alpha particle and the gold nucleus – **Exercise!** We shall come back to this question in our lectures on nuclear physics.

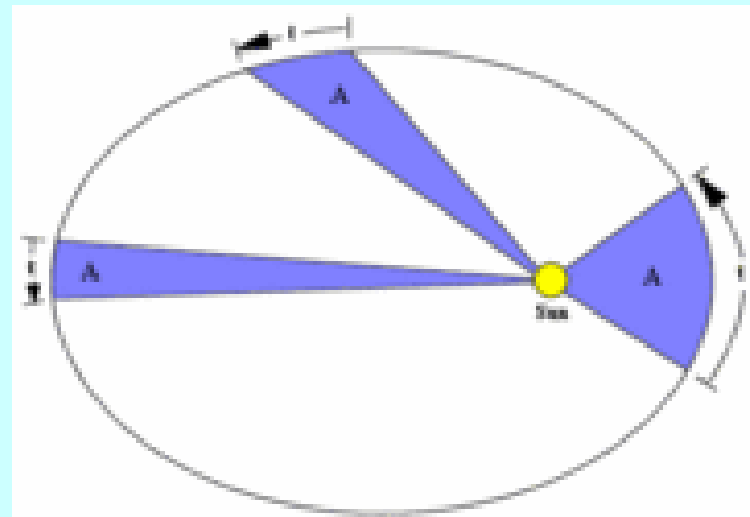


perihelion: the vertex nearest the sun.

aphelion: the vertex furthest from the sun.

Kepler's 2nd Law: the radius vector of the planet sweeps equal areas in equal intervals of time.

Kepler's 1st Law: a planet moves on an ellipse with the sun standing in one of the foci of the ellipse.



Tides

In the discussion of tides it will be convenient to use the concept of a *potential*.

Thus consider the *potential energy* of a particle of mass m in the gravitational field of a mass M located at the origin:

$$V(r) = -\frac{GmM}{r} = m\Phi(r)$$

This defines the *gravitational potential* Φ of M at r :

$$\Phi(r) = -\frac{GM}{r}$$

If M is at a point with radius vector a , then the gravitational potential at \mathbf{r} is

$$\Phi(\vec{r}) = -\frac{GM}{|\vec{r} - \vec{a}|}$$

If $r/a < 1$, then we can expand the potential in powers of r/a :

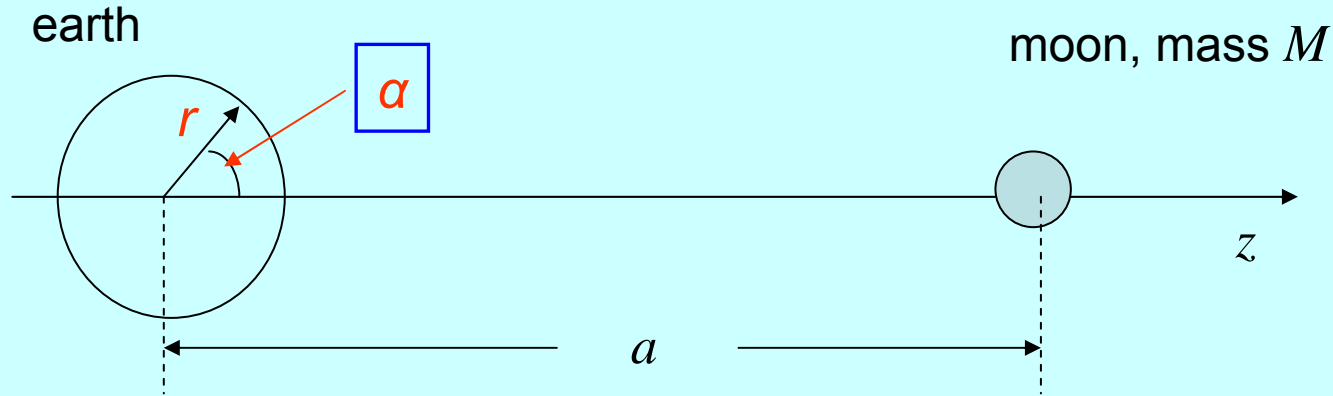
$$\Phi(\vec{r}) = -\frac{GM}{a} \left[1 + \frac{r}{a} \mu + \frac{1}{2} \left(\frac{r}{a} \right)^2 (3\mu^2 - 1) + \mathcal{O} \left(\left(\frac{r}{a} \right)^3 \right) \right] \quad (4.18)$$

where $\mu = \cos \alpha$, and α is the angle between \vec{a} and \vec{r}

Thus we can find the gravitational potential on the surface of the earth due to the moon.

Let the centre of the moon lie on the z axis at a distance a from the centre of the earth, and let vector \mathbf{r} be the radius vector of a point ... 19

... on the surface of the earth: see figure



The gravitational field at \vec{r} is defined by

$$\vec{g}(\vec{r}) = -\nabla\Phi(\vec{r})$$

$$\begin{aligned}\vec{g}(\vec{r}) &= -\nabla \Phi(\vec{r}) = \frac{GM}{a} \nabla \left[1 + \frac{r}{a} \mu + \frac{1}{2} \left(\frac{r}{a} \right)^2 (3\mu^2 - 1) + \mathcal{O} \left(\left(\frac{r}{a} \right)^3 \right) \right] \\ &= \frac{GM}{a} \left\{ \nabla \left(\frac{r}{a} \mu \right) + \frac{1}{2} \nabla \left[\left(\frac{r}{a} \right)^2 (3\mu^2 - 1) \right] + \mathcal{O} \left(\nabla \left(\frac{r}{a} \right)^3 \right) \right\}\end{aligned}$$

Now $r\mu = z$, hence $\nabla(r\mu) = (0, 0, 1) = \hat{z}$

and that means that the first term contributes a constant force in z direction which is the same for all parts of the earth: this is the overall gravitational attraction of the earth by the moon.

In the second term we have

$$\begin{aligned}\left(\frac{r}{a}\right)^2 (3\mu^2 - 1) &= \frac{1}{a^2} (3r^2 \mu^2 - r^2) \\ &= \frac{1}{a^2} [3z^2 - (x^2 + y^2 + z^2)] = -\frac{1}{a^2} (x^2 + y^2 - 2z^2)\end{aligned}$$

and hence the ***tidal field***:

$$\begin{aligned}\vec{g}_t(\vec{r}) &= \frac{GM}{2a} \nabla \left[\left(\frac{r}{a}\right)^2 (3\mu^2 - 1) \right] = -\frac{GM}{2a^3} \nabla (x^2 + y^2 - 2z^2) \\ &= -\frac{GM}{a^3} (x, y, -2z)\end{aligned}$$

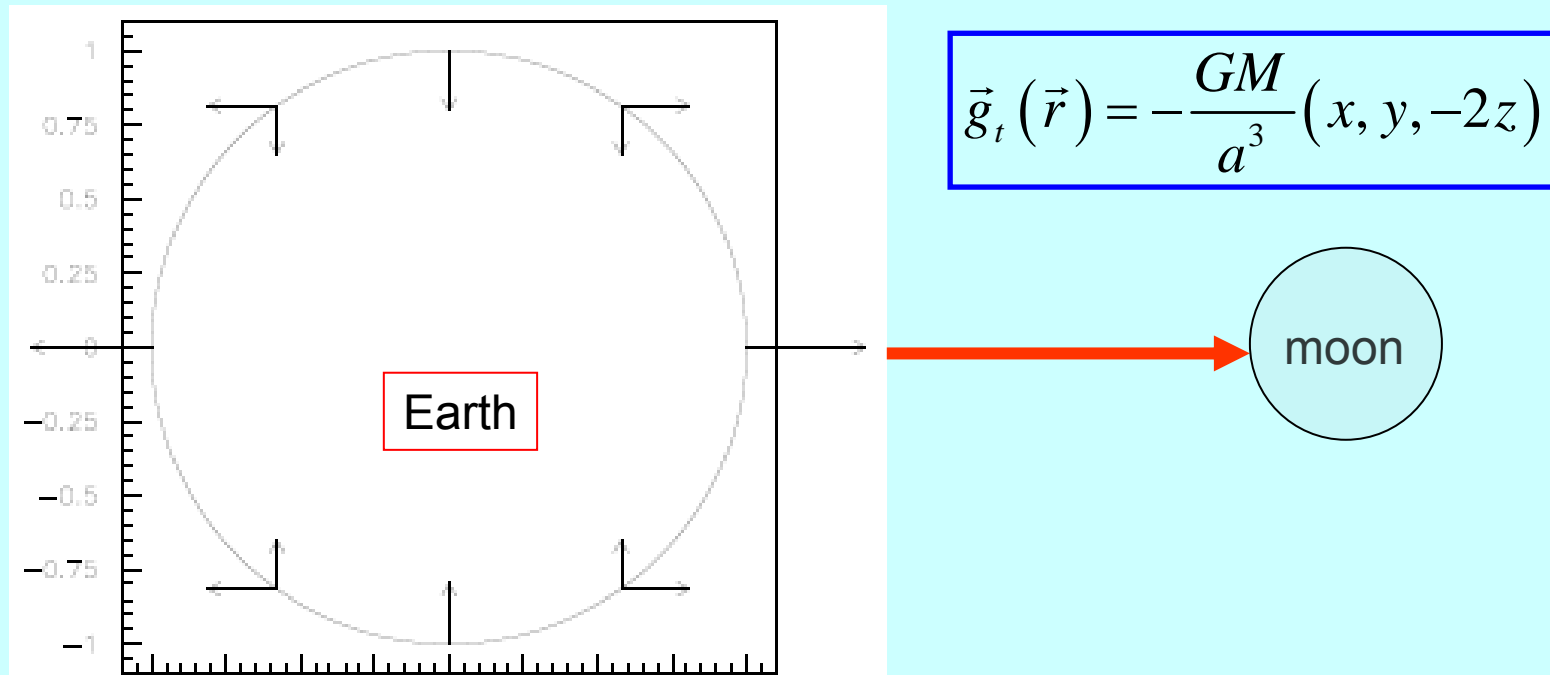
Here, for reference, again the final result:

$$\vec{g}_t(\vec{r}) = -\frac{GM}{a^3}(x, y, -2z)$$

Discussion (see also the figure on the next slide):

- 1.) If $x = y = 0$, then there is a tidal force **towards** the moon for $z > 0$ and **away from** the moon at $z < 0$;
- 2.) If $z = 0$, then the tidal force is directed **towards the centre of the earth**;
- 3.) **[Exercise!]** find the polar angle θ at which the **radial component** of the tidal force is equal to zero.

At this angle the tidal force is tangential to the earth's surface, giving rise to tidal currents. These tidal currents lead to large differences in the levels of the sea surface near the coast, especially in river mouths.



In the figure, tidal forces are shown on the earth's surface caused by the gradient of the lunar gravitational potential.

There are also tidal forces caused by the field of the **sun**: these are by magnitude about half of the tidal forces caused by the moon.

They **enhance** the lunar tidal forces when earth, sun and moon are **aligned** and **diminish** them when the moon is in the **first** or **third quarter**.

Compare the tidal forces on the earth's surface

$$\vec{g}_t(\vec{r}) = -\frac{GM}{a^3}(x, y, -2z)$$

with the gravitational attraction of the earth:

$$g(R) = \frac{GM_E}{R^2}$$

The tidal force has its maximum when $x = y = 0$ and $z = R$ [**Exercise!**]:

$$g_t^{\max} = \frac{2GMR}{a^3}$$

The ratio of the (maximum!) tidal force to the earth's attraction is

$$\frac{g_t^{\max}}{g} = \frac{2M}{M_E} \left(\frac{R}{a} \right)^3$$

Now,

$$M/M_E \approx 1/80 \quad \text{and} \quad R/a \approx 1/60$$

hence

$$g_t^{\max} / g \approx 1.2 \times 10^{-7}$$

Similarly we get for the solar tidal force

$$g_{tSun}^{\max} / g \approx 0.6 \times 10^{-7}$$

and we see that the tidal force of the sun is half that of the moon.