

Selected Topics in Physics

a lecture course for 1st year students

by W.B. von Schlippe

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Lecture 5

Relativity

2. Special Relativity

The principle of relativity was first clearly understood by **Galileo** (1564-1642). It means that experiments equally set up in inertial frames of reference which are moving relative to each other yield equal results.

Einstein postulated that **the principle of relativity applies to all natural phenomena**, and not only to mechanical ones:

this is the 1st postulate of special relativity.

Einstein also took account of the empirical evidence of the independence of the speed of light from the relative motion of source and observer; he made this the **2nd postulate of special relativity**

From these postulates he deduced the **Lorentz transformation**, *i.e.* the transformation which leaves the equations of electromagnetism unchanged.

The ***Lorentz transformation*** is of the following form:

$$\begin{aligned}x &\rightarrow x' = \gamma(x + Vt), \\t &\rightarrow t' = \gamma\left(t + \frac{V}{c^2}x\right), \\y &\rightarrow y' = y, \\z &\rightarrow z' = z\end{aligned}$$

where it is assumed that the primed and unprimed frames coincide at time $t = 0$ (and hence $t' = 0$) and move with relative velocity V in x direction; c is the speed of light in vacuum and γ is given by

$$\gamma = 1 / \sqrt{1 - \frac{V^2}{c^2}}$$

For all everyday speeds, γ differs from 1 by an immeasurably small amount, and the Lorentz transformation turns into the Galilean transformation.

To get a picture of the transformations, both Galilean and Lorentz, it is useful to introduce the concept of a *world line*:

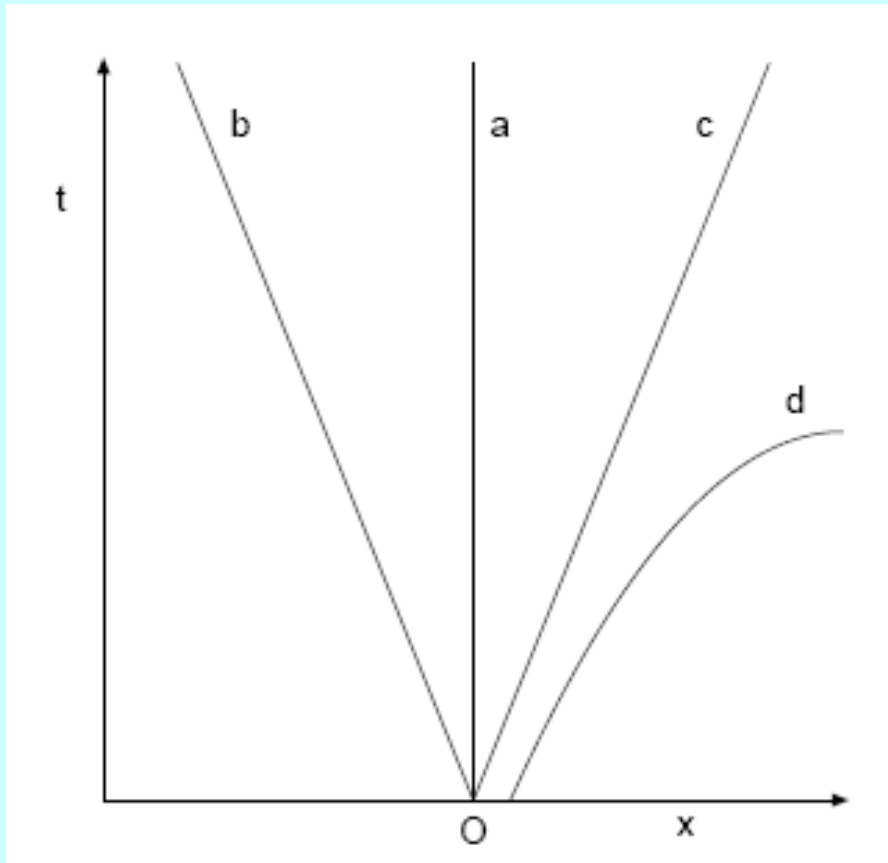
A world line is the line traced out by a body in the course of time.

Thus if a pointlike object is travelling along the x axis with a constant velocity V , then the equation of its world line (in the (x, t) plane!) is

$$x = x_0 + V(t - t_0)$$

assuming that its position is x_0 at time t_0 .

Examples of world lines are shown in the next figure.



World lines:

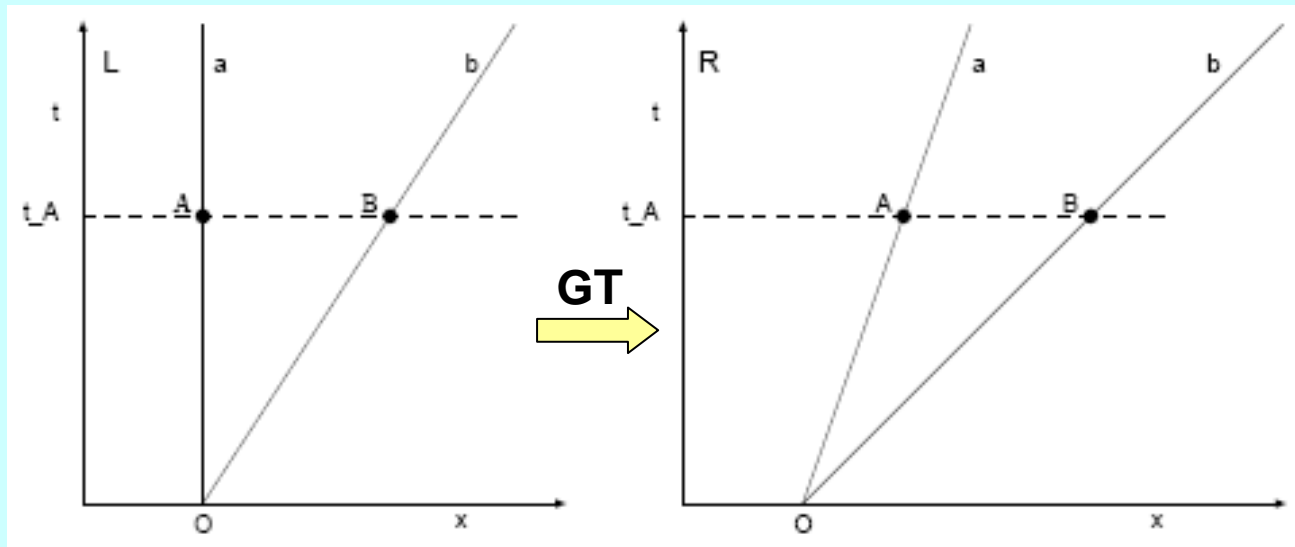
- a) point at rest at the origin;
- b) point moving left with constant speed;
- c) point moving right with constant speed;
- d) world line of accelerated motion.

*only one spatial dimension is shown;
time flows vertically upwards.*

World lines help us to visualise transformations between inertial frames. This is shown in the following figures.

Galilean transformation.

Straight world lines transform into straight world lines
(*inertial frames transform into inertial frames!*)



L: world line a: point at rest;
world line b: point moving
with constant speed

R: after Galilean transformation

$$x \rightarrow x' = x + Vt$$

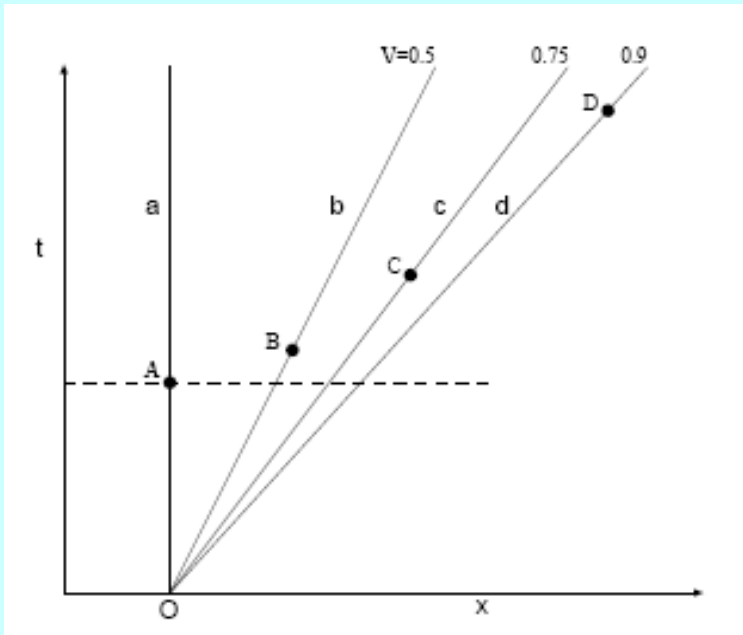
A point in the (x, t) plane is called an event.

Thus a world line is a continuous sequence of events.

Events A and B are
simultaneous

Events A and B remain
simultaneous

World lines of a point at rest in the (x, t) frame (a) as seen by an observer travelling with velocity $V=0.5 c$ (b); $0.75 c$ (c); and $0.9 c$ (d).



Event A appears to these observers at B, C, and D, respectively.

Lorentz transformation:

$$x \rightarrow x' = \gamma (x + Vt)$$

$$t \rightarrow t' = \gamma \left(t + \frac{V}{c^2} x \right)$$

$$\gamma = 1 / \sqrt{1 - V^2 / c^2}$$

where it is assumed that the inertial frames coincide at $t = t' = 0$.

(in the figure the speed of light is put equal to 1; alternatively you can say that the ordinate is ct , and not just t)

Discussion of the 2nd postulate of special relativity.

The speed of light, measured by an observer in an inertial frame, does not depend on the speed with which he is moving relative to the source of light.

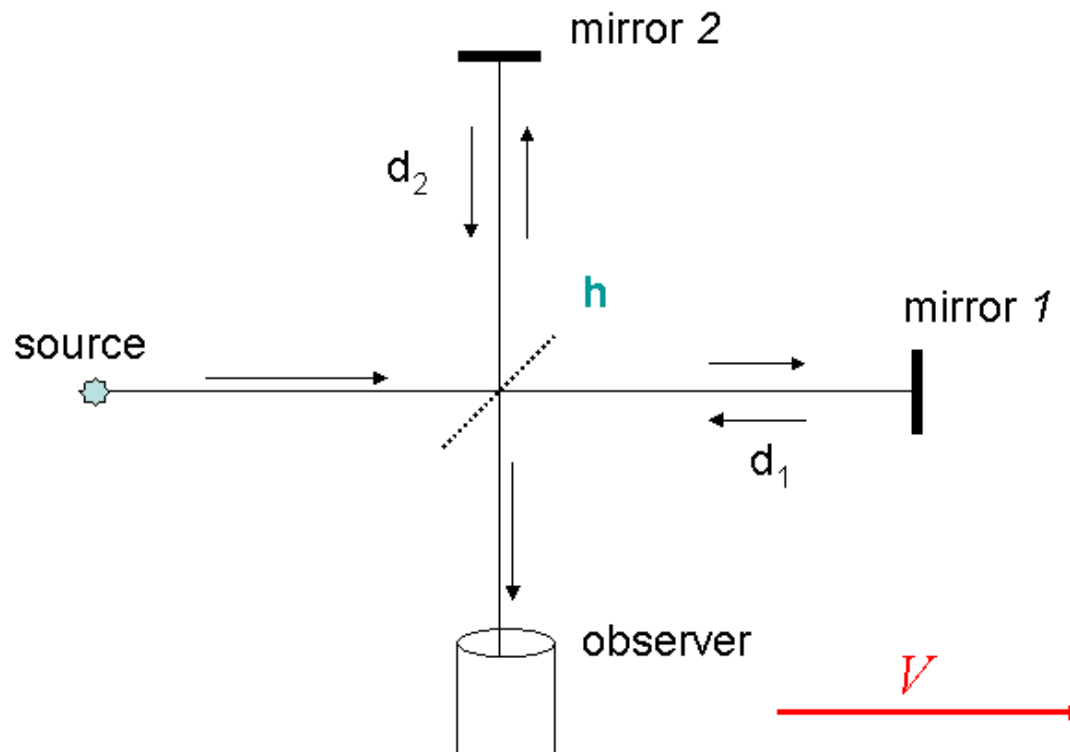
This is different from the measured speed of sound!

Sound is carried by elastic media, for instance by air. If the observer is moving towards the source in still air, then he finds that the sound propagates with a speed that is the sum of the speed of sound in still air and his own speed.

Before Einstein it was similarly assumed that light was carried by an elastic medium, the *ether*. It was therefore conceivable (*natural!*) that the speed of light measured by observers in different inertial frames should be different.

This assumption was first tested experimentally by **Michelson** and **Morley** with the result that no such effect could be detected.

Michelson interferometer



h: half-silvered mirror ("*beam splitter*")

$d_{1,2}$: distance of mirror 1,2 from beam splitter

V : velocity relative to the hypothetical ether

Theory of the Michelson-Morley Experiment.

Assume that light is carried by the ether with velocity c (*relative to the ether!*).

Assume that arm 1 of the interferometer is pointing in the direction of motion of the interferometer relative to the ether, travelling with velocity u .

Then the time taken by the light from h to mirror 1 is

$$t_1^d = \frac{d_1}{c - u}$$

and the time taken by the reflected light is

$$t_1^r = \frac{d_1}{c + u}$$

and hence the total time from h to mirror 1 and back to h is:

$$t_1 = \frac{d_1}{c - u} + \frac{d_1}{c + u} = 2 \frac{d_1}{c} \frac{1}{1 - u^2/c^2}$$

To find the time the ray takes from h to mirror 2 and back we must take account of the distance travelled by the apparatus during this time.

Denote by t_2 the time it takes the ray from A to M_2 and back to B

Then the distance travelled by the apparatus during that time is

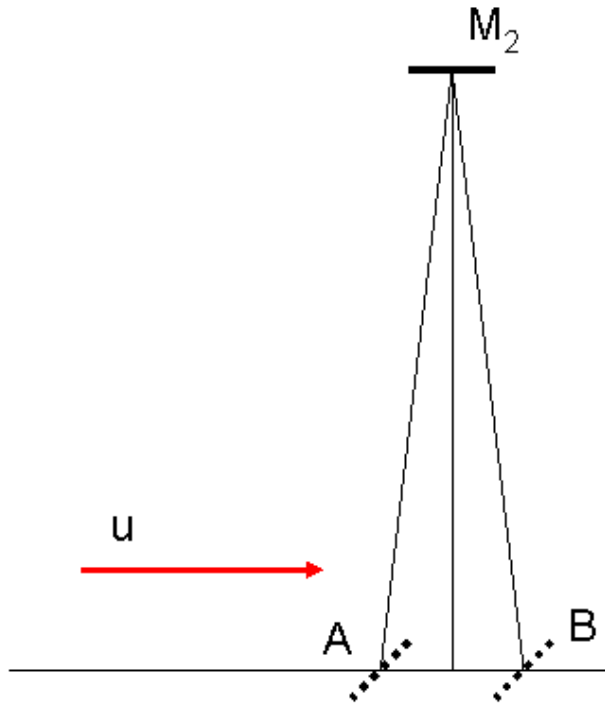
$$l_2 = ut_2$$

and since the length of the arm is d_2 ,
the distance travelled by the ray is

$$L = \sqrt{4d_2^2 + (ut_2)^2}$$

and since it is travelling with velocity c , this is equal to ct_2 , i.e.

$$ct_2 = \sqrt{4d_2^2 + (ut_2)^2}$$



hence solving for t_2 we get

$$t_2 = \frac{2d_2}{c} \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

and therefore the time difference along the two arms is

$$\Delta t_1 = t_2 - t_1 = \frac{2}{c} \left(\frac{d_2}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{d_1}{1 - \frac{u^2}{c^2}} \right)$$

If now the interferometer is rotated through 90 degrees, then the other arm is pointing in the direction of motion, and by a similar calculation we get a time difference ...

$$\Delta t_2 = \frac{2}{c} \left(\frac{d_2}{1 - \frac{u^2}{c^2}} - \frac{d_1}{\sqrt{1 - \frac{u^2}{c^2}}} \right)$$

The expected effect is a shift of the interference pattern resulting from the difference in time $\Delta t_2 - \Delta t_1$:

$$\mathcal{G} = \Delta t_2 - \Delta t_1 = \frac{2}{c} \frac{d_1 + d_2}{\sqrt{1 - \frac{u^2}{c^2}}} \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right)$$

and hence for $(u/c)^2 \ll 1$, keeping only the first nonvanishing term,

$$\mathcal{G} = \frac{d_1 + d_2}{c} \frac{u^2}{c^2}$$

The shift of the fringes must be compared with their width: to give an observable effect this ratio should be not less than 0.1

The shift of the fringes is proportional to the time difference:

$$\Delta s \propto \mathcal{G}$$

and the width of the fringes is inversely proportional to the frequency of light:

$$s \propto \frac{1}{\nu}$$

hence

$$\frac{\Delta s}{s} \sim \frac{\mathcal{G}}{1/\nu} = \frac{\nu}{c} (d_1 + d_2) \frac{u^2}{c^2} = \frac{1}{\lambda} (d_1 + d_2) \frac{u^2}{c^2}$$

where λ is the wave length of light used in the interferometer, *i.e.*

$$\lambda \sim 500 \text{ nm} = 5 \times 10^{-7} \text{ m}$$

Assume that the interferometer is moving relative to the ether with a speed equal to the orbital velocity of the earth, then

$$u \approx 3 \times 10^4 \text{ m s}^{-1}$$

hence

$$u^2/c^2 \approx 10^{-8}$$

and

$$\frac{\Delta s}{s} \sim \frac{2\bar{d}}{5 \times 10^{-7}} \cdot 10^{-8} = \frac{2\bar{d}}{50 \text{ m}}$$

Using multiple reflection, the effective paths d_1 and d_2 can be increased to several meters. It is therefore possible to get a shift of interference fringes of the order of several percents of their width, which should be comfortably observable.

Absence of an effect can of course mean that at the time of the experiment the earth is at rest relative to the ether. Then, repeating the experiment after six months, the earth should have a velocity of 60 km per second relative to the ether. If no effect is seen then either, then one must conclude that the concept of the ether as a carrier of e.m. waves is wrong.

That was Einstein's idea.

Another strong reason for saying that the observed speed of light is independent of the relative motion of source and observer comes from Maxwell's theory of electromagnetism, which gives for the speed of propagation of electromagnetic waves in vacuum the result

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

where ϵ_0 is the (electric) **permittivity** and μ_0 is the (magnetic) **permeability** of free space. These constants do in no way depend on the reference frame in which they are measured.

More empirical evidence comes from astronomical observations of light from double stars and from supernovas. These observations support the statement of Einstein's second postulate of relativity.

Derivation of the Lorentz Transformation from the Postulates of Relativity.

Consider two inertial frames

$$K(x, y, z, t) \quad \text{and} \quad K'(x', y', z', t')$$

We want to find a transformation that transforms straight world lines of K into straight world lines of K'

Such a transformation must necessarily be *linear*.

We also demand that for small velocities the new transformation turns into the Galilean transformation:

The 2nd postulate is used in the following way: let a light flash be set off at the origin of K at time $t = 0$. in free space we see a spherical wave front spreading with velocity c :

$$x^2 + y^2 + z^2 = c^2 t^2 \tag{1}$$

Then, assuming that the two inertial frames coincide at time $t = t' = 0$, a spherical wave front spreading out with velocity c is also seen in K' :

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \quad (2)$$

Then, if we assume that K' moves in x direction with velocity v relative to K , we have

$$x \rightarrow x' = \gamma(x - vt)$$

$$y \rightarrow y' = y$$

$$z \rightarrow z' = z$$

$$t \rightarrow t' = \alpha t + \beta x,$$

We substitute this into (2) and demand that it should be identical with (1):

$$\gamma^2 (x - vt)^2 + y^2 + z^2 = c^2 (\alpha t + \beta x)^2$$

or

$$(\gamma^2 - \beta^2 c^2) x^2 + y^2 + z^2 = (\alpha^2 c^2 - \gamma^2 v^2) t^2 + 2(\alpha \beta c^2 + \gamma^2 v) tx$$

hence comparing with (1) we get

$$\begin{aligned} \gamma^2 - \beta^2 c^2 &= 1 & (3) \\ \alpha^2 c^2 - \gamma^2 v^2 &= c^2 & (4) \\ \alpha \beta c^2 + \gamma^2 v &= 0 & (5) \end{aligned}$$

$$(3): \quad \gamma^2 = 1 + \beta^2 c^2 \quad (6)$$

$$(5): \quad \gamma^2 = -\alpha \beta \frac{c^2}{v} \quad (7)$$

(4)+(5) v :

$$\alpha^2 c^2 + \alpha \beta v c^2 = c^2$$

$$\alpha(\alpha + \beta v) = 1 \quad (8)$$

(3) v -(5):

$$\beta(\alpha + \beta v) = -\frac{v}{c^2} \quad (9)$$

(9)/(8):

$$\beta = -\alpha \frac{v}{c^2} \quad (10)$$

(10)→(8):

$$\alpha \left(\alpha - \alpha \frac{v^2}{c^2} \right) = 1$$

$$\alpha^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

Sign of α : for $v \rightarrow 0$ we get $\alpha^2 \rightarrow 1$ and hence $\beta \rightarrow 0$, i.e. $t' \rightarrow \alpha t$, and we must demand in this limit $t' \rightarrow t$, i.e. $\alpha > 0$, thus

$$\alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and hence with (10):

$$\beta = -\alpha \frac{v}{c^2} = -\frac{v/c^2}{\sqrt{1 - v^2/c^2}}$$

and with (6):

$$\gamma^2 = 1 + \beta^2 c^2 = 1 + \frac{v^2/c^2}{1 - v^2/c^2} = \frac{1}{1 - v^2/c^2}$$

Sign of γ : for $v/c \ll 1$ we must have the Galilean transformation:

$$x \rightarrow x' = \gamma(x - vt) \Big|_{v/c \ll 1} \rightarrow x - vt$$

hence $\gamma > 0$, thus

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Summary: the Lorentz transformation is

$$\begin{aligned}x &\rightarrow x' = \gamma(x - vt) \\y &\rightarrow y' = y \\z &\rightarrow z' = z \\ct &\rightarrow ct' = \gamma\left(ct - \frac{v}{c}x\right),\end{aligned}$$

The inverse transformation from K' to K is obtained by exchanging the primed and unprimed coordinates and changing the sign of v (Exercise!):

$$x' \rightarrow x = \gamma(x' + vt')$$

$$y' \rightarrow y = y'$$

$$z' \rightarrow z = z'$$

$$ct' \rightarrow ct = \gamma\left(ct' + \frac{v}{c}x'\right),$$

Lorentz contraction.

consider a rod at rest on the x axis of K ; its end points are x_1 and x_2 , and hence its length is

$$\ell = x_2 - x_1$$

We want to know what its length is in K' .

To measure the length of the rod we must take the readings of the positions of its end points **simultaneously** in the frame, in which the length is to be found, *i.e.* simultaneously in K' . To do this we will be guided by the Lorentz transformation. For end point x_1 we have

$$x_1 \rightarrow x'_1 = \gamma(x_1 - vt_1), \quad ct_1 \rightarrow ct'_1 = \gamma\left(ct_1 - \frac{v}{c}x_1\right),$$

and a similar pair of equations holds for x_2 . Simultaneity in K' ...

means

$$t'_1 = t'_2$$

hence

$$t'_2 - t'_1 = \gamma \left(t_2 - t_1 - \frac{v}{c^2} (x_2 - x_1) \right) = 0$$

and this gives us

$$t_2 - t_1 = \frac{v}{c^2} (x_2 - x_1) = \frac{v}{c^2} \ell$$

and therefore the length of the rod as measured in K' is

$$\begin{aligned} \ell' &\equiv x'_2 - x'_1 = \gamma (x_2 - x_1 - v(t_2 - t_1)) \\ &= \gamma \left(1 - \frac{v^2}{c^2} \right) \ell = \frac{1}{\gamma} \ell \end{aligned}$$

or finally

$$\ell' = \frac{1}{\gamma} \ell = \sqrt{1 - \frac{v^2}{c^2}} \ell$$

Thus we have the result that the length of the rod as measured in a frame that is moving relative to the rod is shortened by an amount, given by the relativistic gamma factor. This is called **Lorentz contraction**.

The length of the rod in its rest frame is called its *proper length*. Therefore we can also express our result by saying that the length measured in a moving frame is less than the proper length.

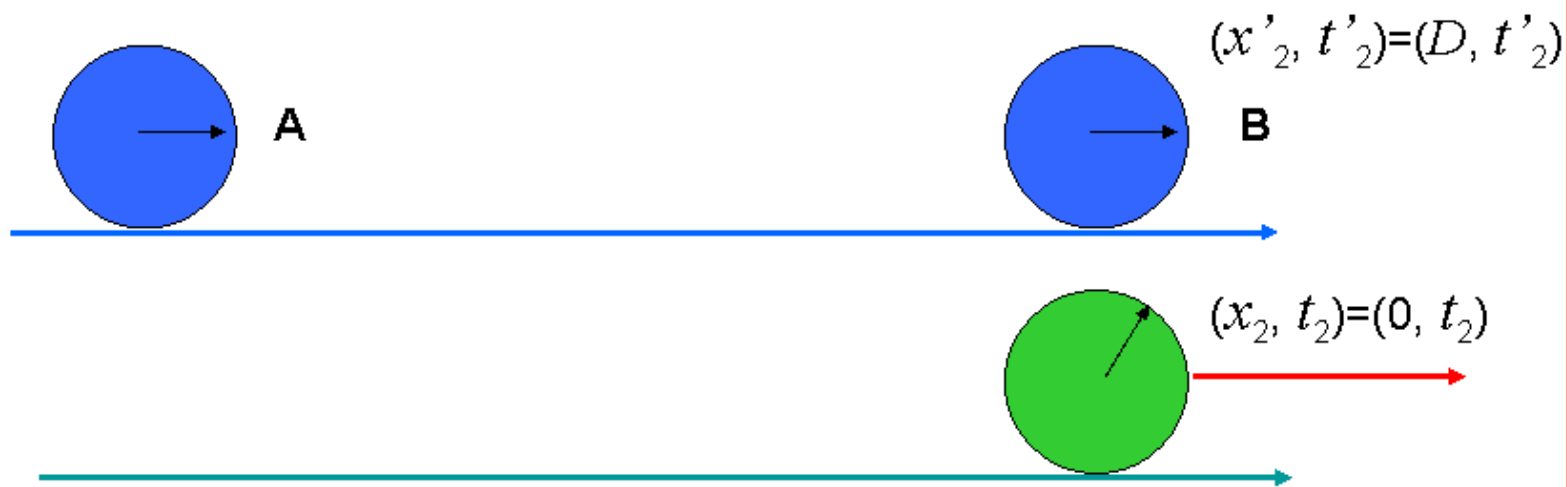
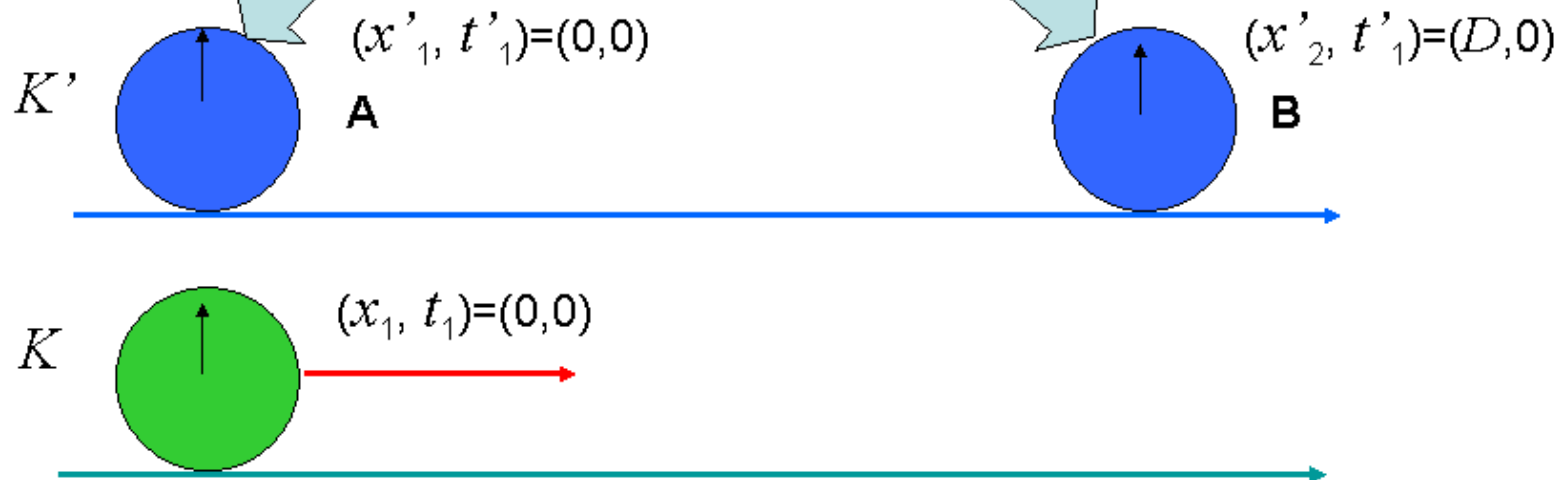
Einstein time dilation.

Synchronisation of clocks: clocks within one reference frame can be synchronised by the exchange of signals if the speed is known with which the signals propagate.

Clocks in frames moving relative to each other can be synchronised when they are at the same point in space.

This is illustrated in the next figure.

Clocks in K' are synchronised



The green clock is at rest in K . When its position coincides with clock **A** of K' , it is synchronised with **A**. At some later time it is passing clock **B** and the readings of the clocks are compared.

Thus we have:

$$x_2 = \gamma (x'_2 + vt'_2) = 0$$

$$t_2 = \gamma \left(t'_2 + \frac{v}{c^2} x'_2 \right),$$

From the former equation we find

$$x'_2 = -vt'_2$$

substitution of which into the latter equation gives

$$t_2 = \gamma \left(t'_2 - \frac{v^2}{c^2} t'_2 \right) = \frac{1}{\gamma} t'_2$$

or

$$t_2' = \gamma t_2$$

The time t_2 measured in the rest frame of the green clock is its proper time; the time t_2' is its time as seen by a moving observer.

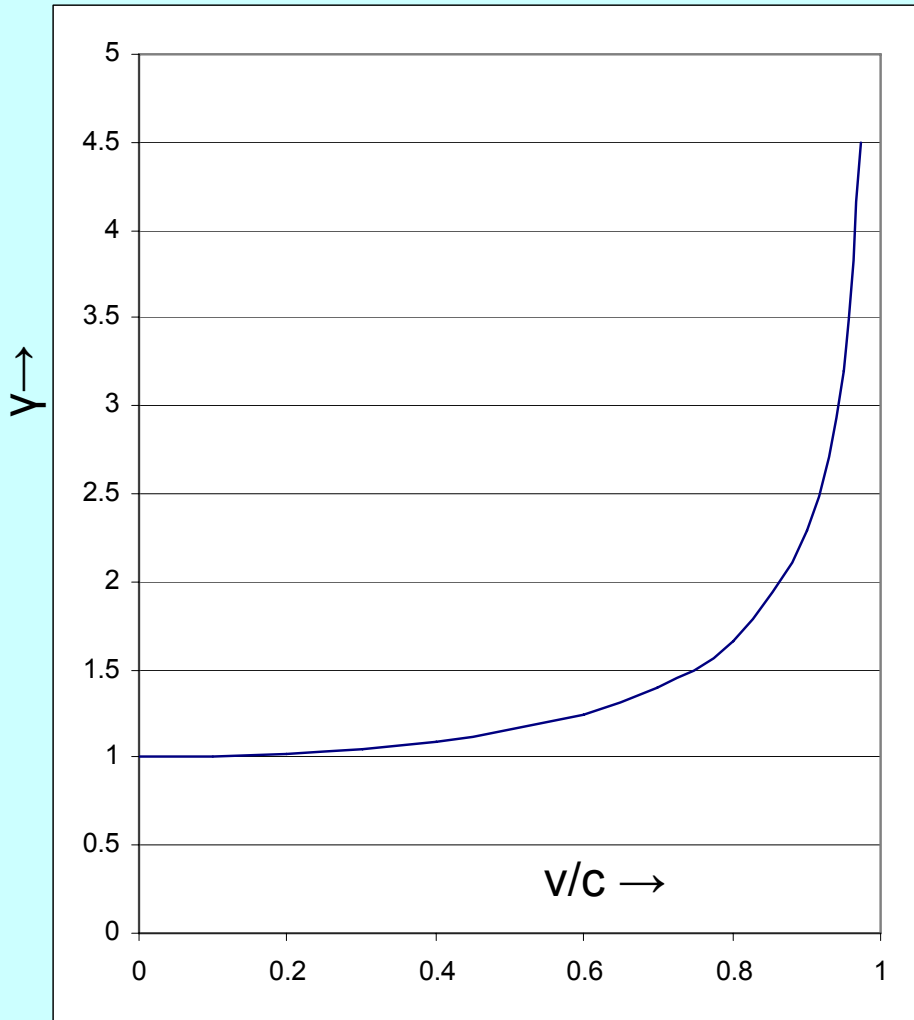
We therefore have the following result:

an observer who is moving relative to a clock measures a time that is greater than the proper time of that clock.

(Einstein time dilation)

Lorentz contraction and time dilation are not noticed in ordinary life. We see that the difference between relativistic physics and everyday experience is expressed by the relativistic gamma factor.

In the next figure the gamma factor is shown as a function of v/c .



We see that below 20% of the speed of light there is practically no difference between Galilean-Newtonian and relativistic physics.

A supersonic aeroplane travels with a velocity of the order of 0.1% of the speed of light: clearly not enough to notice relativistic effects.

Above 20% the difference grows rapidly. Speeds close to the speed of light are common place in high energy physics. Let us look at an example taken from that branch of physics.

In modern high energy accelerators, protons are accelerated to velocities up to 0.999 999 5 times the speed of light. This corresponds to a gamma factor of the order of 1000.

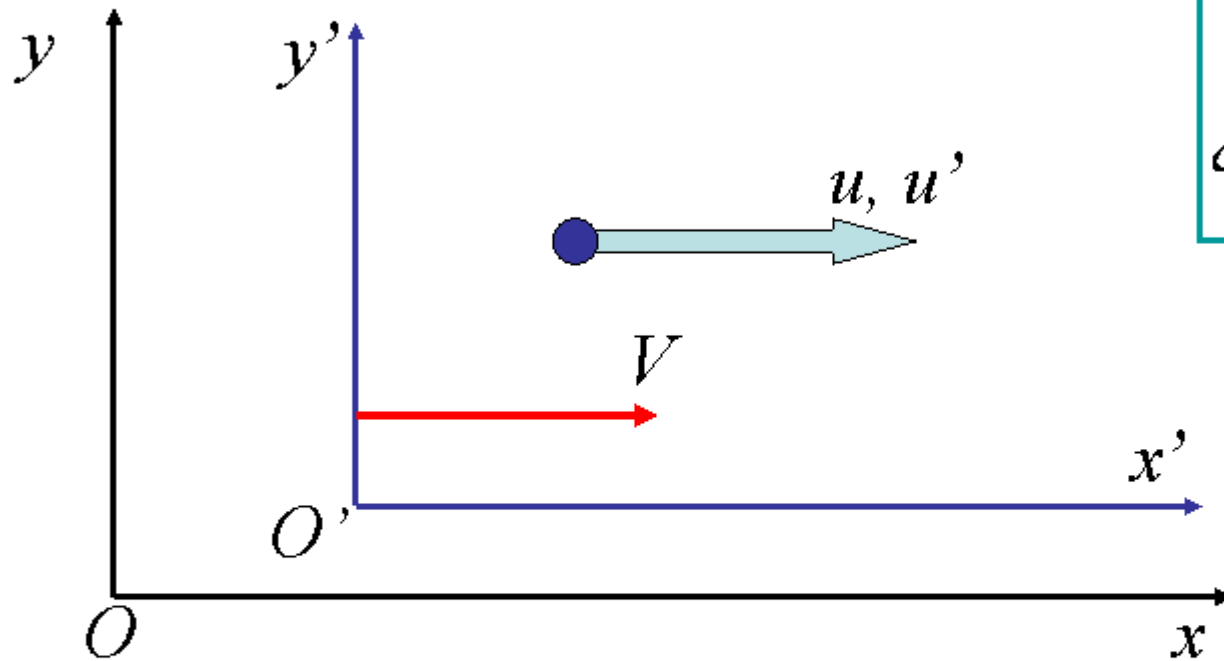
Assume that one of these protons leaks out of the laboratory and travels towards the moon.

The distance between earth and moon is roughly 60 times the earth's radius or about 4×10^8 m. Since the proton travels with practically the speed of light, its journey time is about 1.3 seconds. If that proton were to carry a stopwatch, it would measure a proper time of 1.3 *milliseconds*.

If that proton missed the moon and carried on travelling towards the sun, then it would arrive there in about 8 minutes or roughly 500 seconds, but its proper time for that journey would be only $\frac{1}{2}$ of a second.

More serious than mental games like that are the implications of relativity for the design of high energy accelerators and detectors. But we need some more preparation before we'll be able to talk sensibly about those.

Relativistic addition of velocities



$$\begin{aligned}x' &= \gamma(x - Vt); \\ ct' &= \gamma\left(ct - \frac{V}{c}x\right)\end{aligned}$$

$$u' = \frac{dx'}{dt'} = \frac{dx - Vdt}{dt - (V/c^2)dx} = \frac{u - V}{1 - uV/c^2}; \quad u = \frac{u' + V}{1 + u'V/c^2}.$$

$$u_{|u' \rightarrow c} \rightarrow c$$

Invariance of lengths of intervals.

Consider two events: A and B

Their coordinates are

$$(x_A, y_A, z_A, t_A) \quad \text{and} \quad (x_B, y_B, z_B, t_B)$$

respectively.

We define the interval that separates the events:

$$(x_A - x_B, y_A - y_B, z_A - z_B, t_A - t_B)$$

then one can show that the ***Lorentz invariant square of the interval*** is given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$$

where

$$\Delta x = x_A - x_B, \Delta y = y_A - y_B, \Delta z = z_A - z_B, \Delta t = t_A - t_B$$

Proof: assume for simplicity that we have

$$x \rightarrow x' = \gamma(x - vt),$$

$$y \rightarrow y' = y,$$

$$z \rightarrow z' = z$$

$$ct \rightarrow ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

hence

$$\begin{aligned}
(\Delta s')^2 &= (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - (c\Delta t')^2 \\
&= \gamma^2 (\Delta x - V\Delta t)^2 + (\Delta y)^2 + (\Delta z)^2 - \gamma^2 \left(c\Delta t - \frac{V}{c}\Delta x \right)^2 \\
&= \gamma^2 \left\{ \left(1 - \frac{V^2}{c^2} \right) (\Delta x)^2 - \left(1 - \frac{V^2}{c^2} \right) (c\Delta t)^2 - 2 \left[(V) - (V) \right] \Delta t \Delta x \right\} \\
&\quad + (\Delta y)^2 + (\Delta z)^2
\end{aligned}$$

but

$$\gamma^2 = 1 / \left(1 - \frac{V^2}{c^2} \right)$$

hence

$$(\Delta s')^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2 = (\Delta s)^2$$

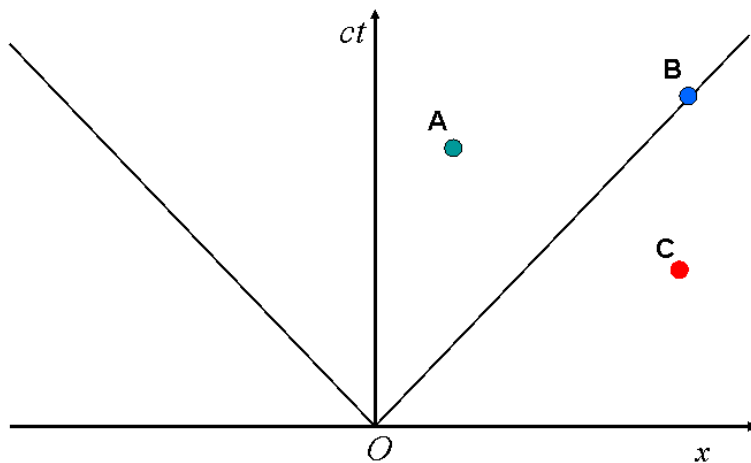
i.e. the square of the interval is Lorentz invariant.

The square of the interval between two events can be positive, negative or zero:

$(\Delta s)^2 > 0$: "space-like separation of events"

$(\Delta s)^2 = 0$: "light-like separation of events"

$(\Delta s)^2 < 0$: "time-like separation of events"



OA: time-like
OB: light-like
OC: space-like

Signals from **O** can reach **A** and **B** but not **C**, therefore a causal relation is possible only between events **A** and **O** or between **B** and **O** but not between **C** and **O**