Selected Topics in Physics

a lecture course for 1st year students

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Spring Semester 2007

Lecture 8

1. Relativistic Kinematics (continued)

2. The Bohr atom

- In the previous lecture we have discussed the kinematics of particle collisions at relativistic energies.

- We have made the distinction between elastic and inelastic collisions and derived several formulas relating energies, momenta and scattering angles in a reference frame in which one of the particles was initially at rest.

This is called Laboratory Frame or simply LAB frame.

- Now I will introduce another frame, the Centre-of-mass Frame or CMS, and then go on to point out formulas relating LAB variables to CMS variables. Their derivation will be left as an exercise.

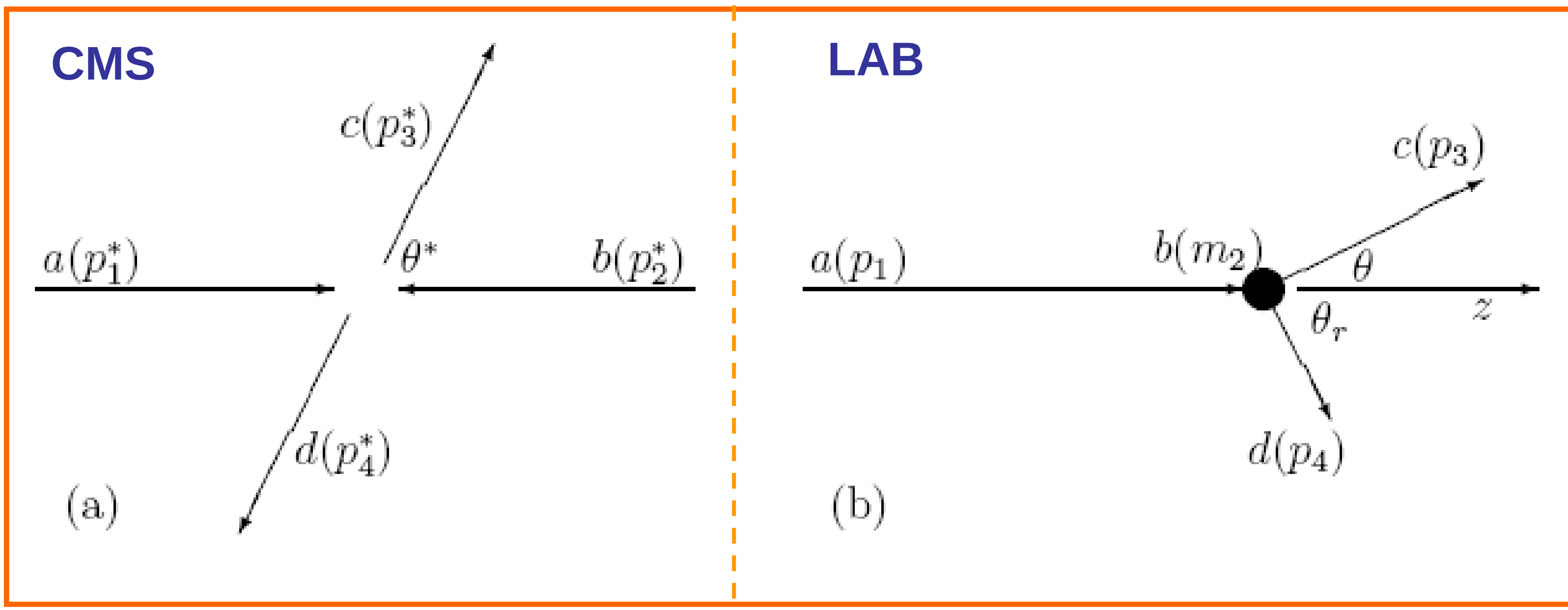
- Then I shall discuss inelastic collisions.

- Given enough time I will then discuss the hydrogen spectrum and its explanation based on the picture of the atom as given by Rutherford and Bohr.

Laboratory and centre-of-mass frames

The reference frame that has a target particle initially at rest is called laboratory frame (**LAB**).

For many theoretical investigations it is more convenient to work in the centre-of-mass frame (**CMS**) which is defined by the momenta of the initial particles being equal in magnitude and opposite in direction.



I will label all **CMS** variables by an ***asterisk* (*)**.

Thus the 4-momenta for the case of elastic scattering $a + b \to a + b$ are in the **CMS** and **LAB**:

CMS

$$\mathrm{p}_a^* = \left(E_a^*, 0, 0, p^*\right),$$
$$\mathrm{p}_b^* = \left(E_b^*, 0, 0, -p^*\right),$$
$$\mathrm{p}_a'^* = \left(E_a'^*, \vec{p}'^*\right),$$
$$\mathrm{p}_b'^* = \left(E_b'^*, -\vec{p}'^*\right).$$

LAB

$$\mathrm{p}_a = \left(E_a, 0, 0, p_a\right),$$
$$\mathrm{p}_b = \left(m_b, 0, 0, 0\right),$$
$$\mathrm{p}_a' = \left(E_a', p_{ax}', p_{ay}', p_{az}'\right),$$
$$\mathrm{p}_b' = \left(E_b', p_{bx}', p_{by}', p_{bz}'\right).$$

I have previously defined the invariant s:

$$s = \left(\mathrm{p}_a + \mathrm{p}_b\right)^2$$

and we can see that expressed in **LAB** variables it is given by

$$s = m_a^2 + m_b^2 + 2 m_b E_a$$

and in the **CMS** it is

$$s = \left(E_a^* + E_b^*\right)^2$$

i.e. s **is the square of the total CMS energy**.

Now since

$$E_a^* = \sqrt{m_a^2 + p^{*2}}, \quad E_b^* = \sqrt{m_b^2 + p^{*2}}$$

we can solve for p^*:

$$\boxed{p^* = \frac{1}{2\sqrt{s}}\left\{\left[s - \left(m_a - m_b\right)^2\right]\left[s - \left(m_a + m_b\right)^2\right]\right\}^{1/2}}$$

The velocity of the **CMS** relative to the **LAB** (***"boost velocity"***) is found by writing down the LT formula for the target particle and noting that by definition its **LAB** momentum is zero:

$$p_{bz} = \gamma\left(p_{bz}^* + VE_b^*\right) = 0,$$

hence

$$V = -\frac{p_{bz}^*}{E_b^*} = \frac{p_{az}^*}{E_b^*}$$

where in the last step I have used the definition of the CMS:

$$p_{az}^* = -p_{bz}^*$$

The scattering angles in the **LAB** and **CMS** are related by

$$\tan\theta = \frac{\sin\theta^*}{\gamma\left(\cos\theta^* + V/v^*\right)}; \quad \tan\theta^* = \frac{\sin\theta}{\gamma\left(\cos\theta - V/v\right)}$$

where

$$v^* = p_3^*/E_3^* \quad \text{and} \quad v = p_3/E_3$$

are the velocities of the scattered particle in the LAB and CMS, respectively.

the derivations of all of these relations is left as an **exercise!**

Inelastic collisions

the reaction equation of an inelastic collision is of the following form:

$$a + b \rightarrow c + d + \ldots$$

the reaction can take place if all required ***conservation laws*** are satisfied

(i) conservation of energy

(ii) conservation of momentum

(iii) conservation of charge

(iv) others

if a reaction is allowed by conservation of energy, momentum and charge but is not observed, then there must be another conservation law prohibiting it, possibly ***a new conservation law that we are discovering!***

In this lecture we will consider only the conservation of energy and momentum to keep within the framework of relativistic kinematics.

Examples of inelastic reactions:

$e^+ + e^- \rightarrow 2\gamma$	electron-positron annihilation
$e^+ + e^- \rightarrow \mu^+ + \mu^-$	muon pair creation …
$\pi^- + p \rightarrow \pi^0 + n$	charge exchange scattering …
$\pi^- + p \rightarrow K^- + \Sigma^+$	kaon-sigma production …
$p + p \rightarrow p + p + p + \overline{p}$	antiproton production …

Using conservation of energy and momentum we can find, for example, the minimum energy needed for an inelastic reaction to go.

This minimum energy is called ***threshold energy.***

Calculation of the threshold energy.

Assume that the initial particles have 4-momenta p_a and p_b, and the final particles have 4-momenta p_1, p_2, …, p_n

Then we have by 4-momentum conservation:

$$p_a + p_b = p_1 + p_2 + \ldots + p_n$$

hence

$$s = (p_a + p_b)^2 = (p_1 + p_2 + \ldots + p_n)^2$$

By definition, the total 3-momentum of the system of particles is zero in the CMS. ***Therefore we calculate the r.h.s. in the CMS***:

$$(p_1 + p_2 + \ldots + p_n)^2 = \left(E_1^* + E_2^* + \ldots + E_n^*\right)^2$$

$$= \left(\sqrt{m_1^2 + \vec{p}_1^{\,2}} + \sqrt{m_2^2 + \vec{p}_2^{\,2}} + \ldots + \sqrt{m_n^2 + \vec{p}_n^{\,2}}\right)^2 \geq (m_1 + m_2 + \ldots + m_n)^2$$

and hence

$$s_{thr} = \min s = \left(m_1 + m_2 + \ldots + m_n\right)^2$$

Now since s is invariant, we can evaluate the l.h.s., *i.e.* s for the initial state, in any reference frame we like, for instance in the LAB frame, where we know that

$$s = m_a^2 + m_b^2 + 2m_b E_a$$

hence

$$E_{thr} = \frac{1}{2m_b}\left[\left(m_1 + m_2 + \ldots + m_n\right)^2 - m_a^2 - m_b^2\right]$$

Example: threshold LAB energy of antiproton production in proton-proton collisions

$$p + p \rightarrow p + p + p + \overline{p}$$

$$E_{thr} = \frac{1}{2m_p}\left[\left(4m_p\right)^2 - 2m_p^2\right] = 7m_p$$

the extra energy above the rest energy of the created proton-antiproton pair goes into the kinetic energy of the final particle system in the LAB.

The Bohr Theory of the Hydrogen Atom

1. Historical Background

By the end of the 19th century several series of hydrogen spectral lines have been measured. They were summarised by ***Balmer*** in the form of a simple formula,

$$k = \frac{1}{\lambda} = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right), \quad m > n = 1, 2, 3, \ldots$$

k = wave number, λ = wavelength, R = Rydberg constant

$$R = 109737.3\ \mathrm{cm}^{-1}$$

Wavelengths of hydrogen spectral lines, in nanometers:

n \ m	2	3	4	5	6	7	8	9	
1	121.5	102.5	97.2	94.9	93.7	93.0	92.6	92.3	UV
2		656.1	486.0	433.9	410.1	396.9	388.8	383.4	Vis/UV
3			1874.6	1281.5	1093.5	1004.7	954.3	922.7	IR

To appreciate the achievement of establishing the Balmer formula, we should note the range of wavelength of the hydrogen spectra: only a few lines lie in the visible region, most lines are either in the ***ultraviolet*** or in the ***infrared*** region. That means that different experimental techniques had to be developed to produce and to measure these spectra.

But even with the empirical formula of Balmer, there was no understanding of the hydrogen spectrum until ***Niels Bohr*** succeeded in deriving the Balmer formula from first principles.

Two more independent lines of investigation were required before Bohr's work:

(i) **Max Planck's** discovery of the quantum nature of electromagnetic energy,

and

(ii) **Rutherford's** discovery of the nuclear structure of the atom.

Towards the end of the 19th century there was a serious problem with ***black body radiation***.

A body is called black body if it does not reflect any electromagnetic radiation falling on it. All e.m. radiation falling on a black body is absorbed.

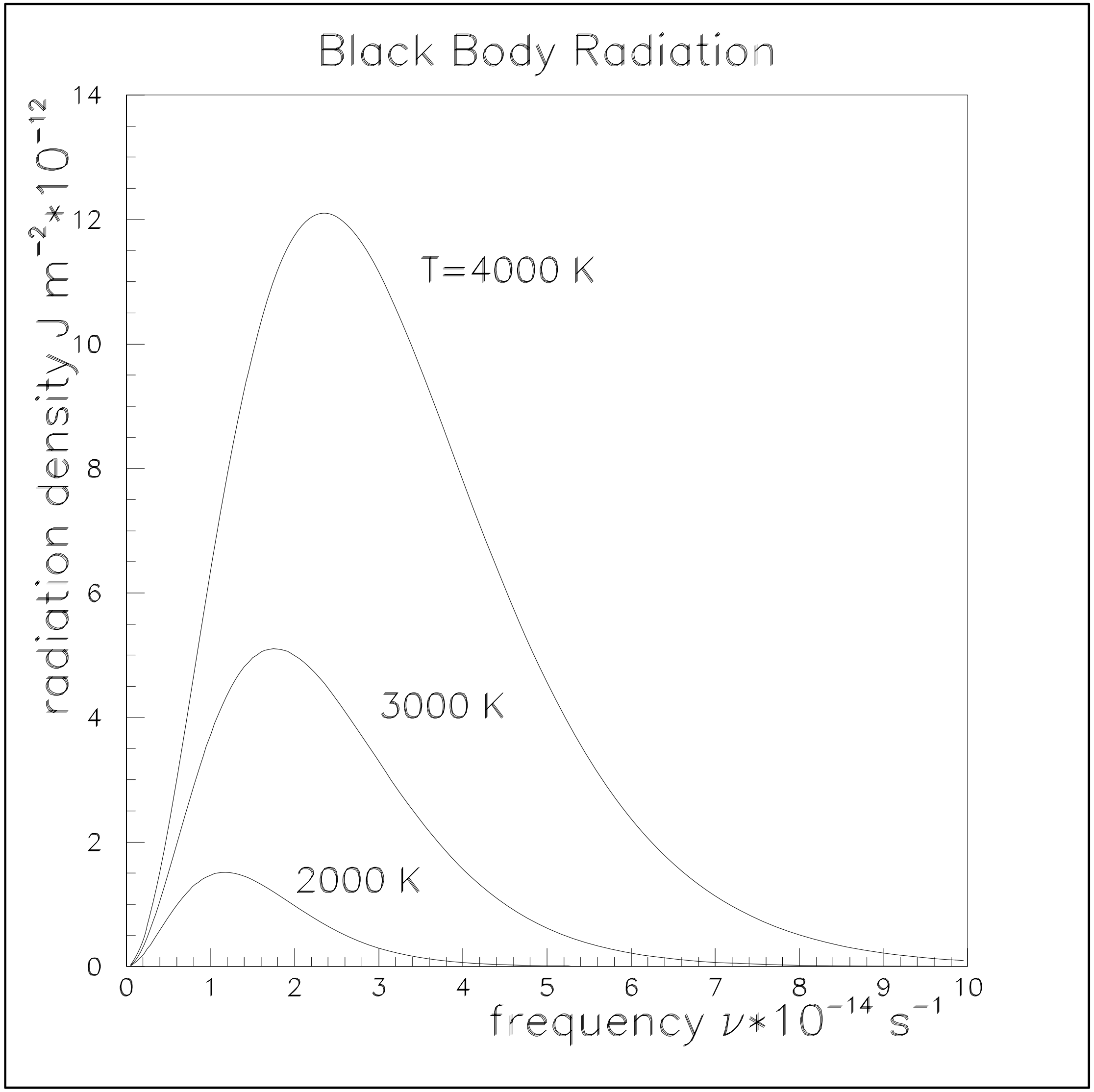
A black body also radiates e.m. energy. In thermal equilibrium it emits as much energy as it absorbs.

Classical electromagnetic theory, applied to the black body, gave agreement only at small frequencies: according to the e.m. theory, the density of radiated energy was proportional to the square of the frequency.

When experimental data became available at high frequencies, a different picture emerged: the radiation density increased, but then the rise slowed down, the density had a maximum and was seen to fall off exponentially.

- In 1900 Max Planck found the formula that describes all data on Black Body radiation.
- To do so he had to demand that radiation can be absorbed and emitted only in *"Portions of Energy"* or quanta. This novel concept was the birth of quantum physics.
- This is Planck's formula:

$$\rho(\nu,T)=\frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}$$



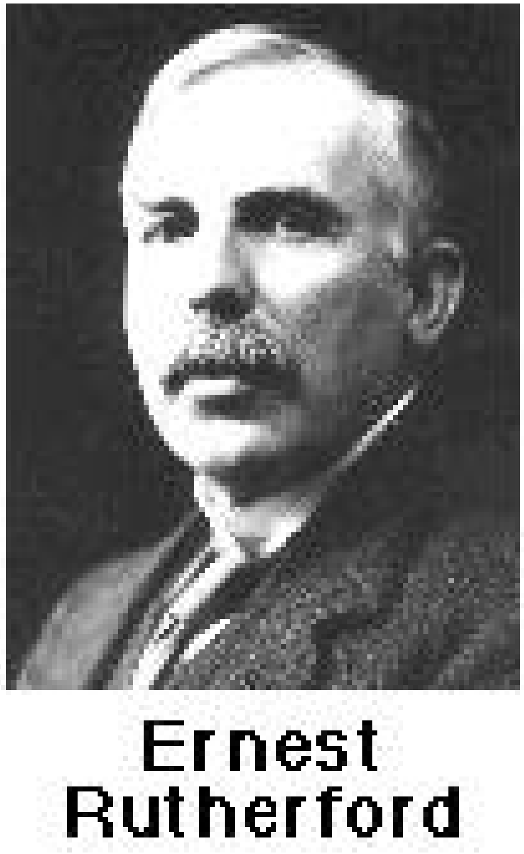
In 1905 Einstein developed Planck’s idea and succeeded in explaining the photoelectric effect.

Einstein proposed that light, as well as having the well known wave property, also had corpuscular property:

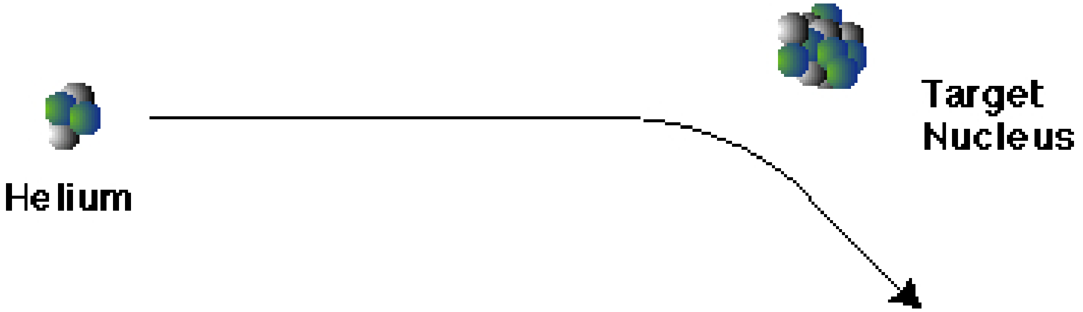
the energy of an electromagnetic wave is not spread out over the entire volume of the wave but is concentrated in a corpuscle of point-like size.

The corpuscles of light were later called ***photons***.

Note that this was a drastic modification of Maxwell’s classical theory of electromagnetism, as indeed has been Planck’s theory of black-body radiation.

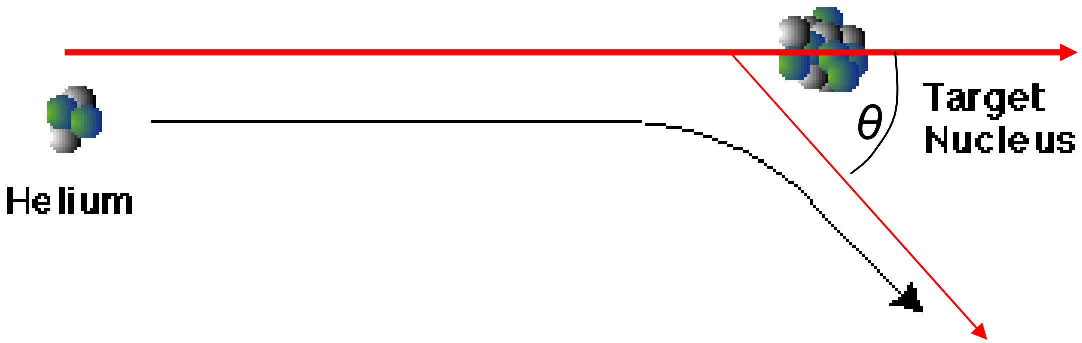


About 100 years ago Ernest Rutherford was experimenting with alpha particles (which he himself had discovered), bombarding gold foils. Comparing his results with the then known theory he discovered the nuclear structure of atoms.



Mathematically, the problem of Rutherford scattering is similar to the Kepler problem that we have discussed in Lectures 3 and 4.

Here I shall retrace some key points of the derivation which will lead us to see quantitatively how close the α particle is coming to the gold nucleus in the Rutherford experiment.



We will see that the trajectory of the alpha particle in the Coulomb field of the gold nucleus is a ***hyperbola***, whose distance of closest approach to the gold nucleus is given by the following formula:

$$r_{\min} = \frac{zZe^2}{2E}\left(1 + \frac{1}{\sin(\theta/2)}\right)$$

All symbols will be explained in due course.

After reduction of the two-body problem (alpha particle and gold nucleus), the equation of motion is

$$\mu\ddot{\vec{r}} = \vec{F}$$

where μ is the reduced mass and F is the Coulomb force,

$$\vec{F} = \frac{zZe^2}{r^2}\hat{r}$$

(ze = charge of the alpha particle, Ze = charge of the nucleus).

As in the Kepler problem we find two conservation laws: conservation of ***energy*** and of ***angular momentum***:

$$\frac{1}{2}\mu\dot{\vec{r}}^2 + V(r) = E = const.$$

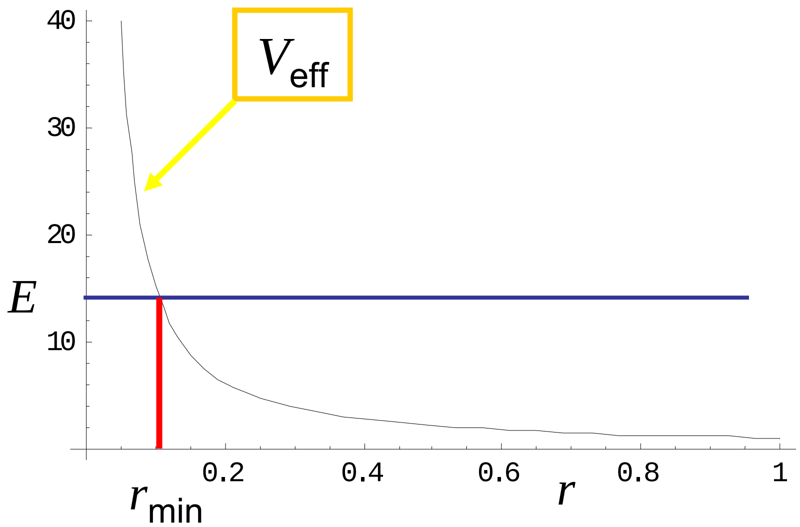
$$\vec{L} = \mu\vec{r}\times\dot{\vec{r}} = const.$$

The kinetic energy is again represented as the sum of two terms,

$$\frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2}$$

and we define again the ***effective potential energy***:

$$V_{eff}(r) = \frac{L^2}{2\mu r^2} + \frac{\gamma}{r}; \quad \gamma = zZe^2$$



The motion is restricted to the region $r > r_{min}$:
r_{min} is the point at which the total energy is equal to the effective *PE*:

.

$$r_{min} = \frac{\gamma}{2E}\left(1 + \sqrt{1 + \frac{2EL^2}{\mu\gamma^2}}\right)$$

The solution of the differential equation goes through as before in Lecture 4, and we get the equation of the trajectory in polar coordinates:

$$r = \frac{\ell}{\varepsilon \cos(\varphi + \alpha) - 1}$$

$$\left(\ell = \frac{L^2}{\gamma\mu}; \quad \varepsilon = \sqrt{1 + \frac{2EL^2}{\gamma^2\mu}}; \quad \alpha \text{ is the integration constant} \right).$$

Since ε >1, the trajectory is a hyperbola.

A hyperbola has two ***asymptotes***;
we choose one asymptote to be parallel to the x axis;
its distance from the x axis is called the ***impact parameter b***

The impact parameter b can be expressed in terms of L and E:

$$b = L\Big/\sqrt{2\mu E}$$

(***Exercise***!)

hence

$$\ell = \frac{2b^2 E}{\gamma}; \quad \varepsilon = \sqrt{1+\left(\frac{\ell}{b}\right)^2}.$$

Our choice of the first asymptote corresponds to $r \to \infty$ for $\varphi \to \pi$, *i.e.*

$$\varepsilon \cos(\pi + \alpha) - 1 = 0$$

hence

$$\boxed{\cos(\alpha) = -\frac{1}{\varepsilon}}$$

The second asymptote corresponds to the ***scattering angle*** θ:

$$\varepsilon \cos(\theta + \alpha) - 1 = 0$$

and after a little algebra we get

$$\boxed{\sin\frac{\theta}{2} = \frac{1}{\varepsilon}}$$

(***Exercise***!)

We can now express the distance of closest approach in terms of E and θ:

$$r_{\min} = \frac{\gamma}{2E}\left(1 + \frac{1}{\sin(\theta/2)}\right)$$

(***Exercise***!)

Recall: $\gamma = zZe^2 = zZ\alpha\hbar c$

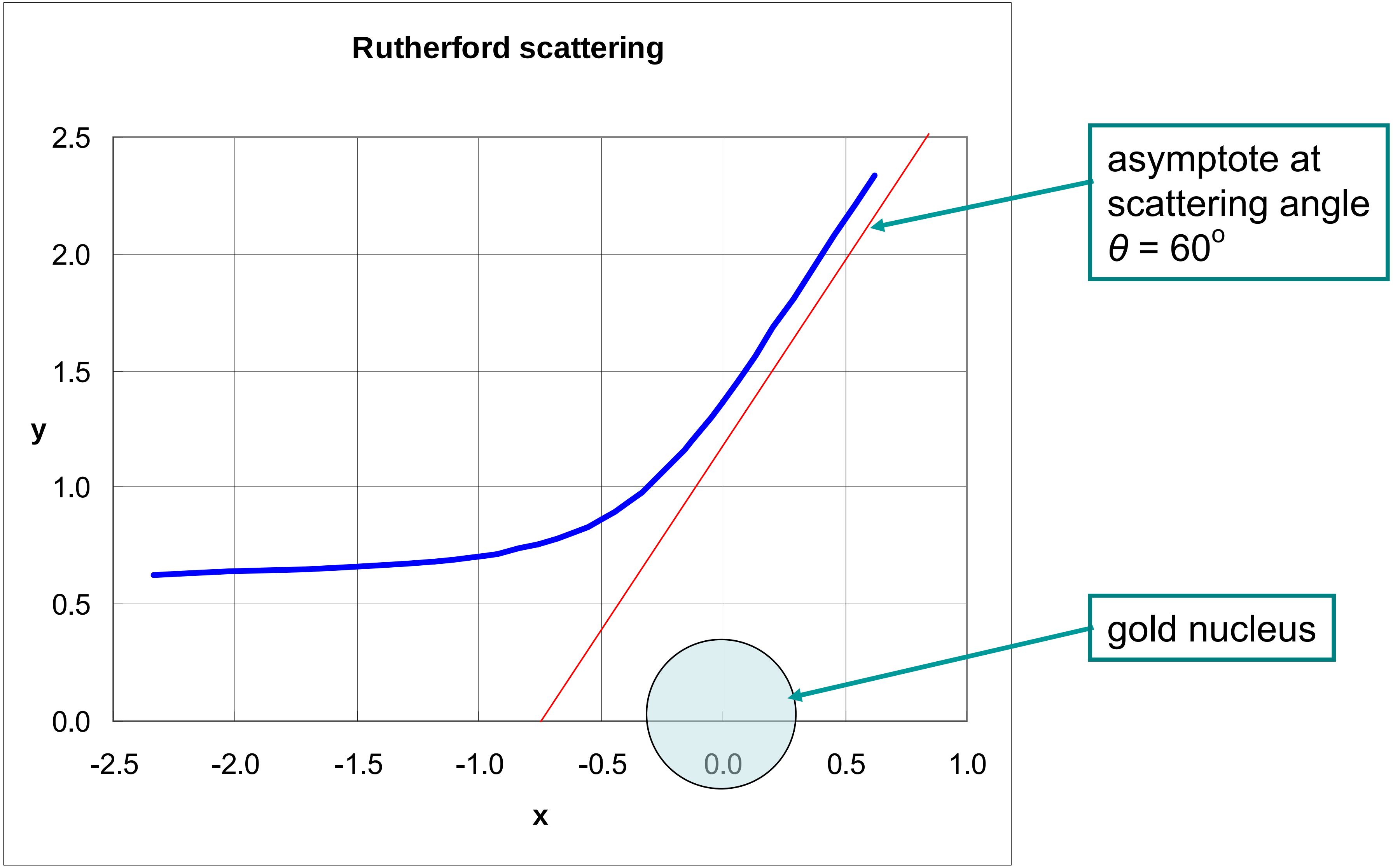
$$\left(\alpha \cong \frac{1}{137}\left(\text{finestructure constant}\right); \quad \hbar c \cong 200\ \text{MeV fm}\right)$$

In the experiment z = 4, Z = 79, and the energy of the alpha particles was a few MeV. Scattering angles as large as 150 degrees were observed.

Assuming a typical energy of E = 5 MeV, we get

$$r_{\min} = \frac{4 \times 79 \times 200}{2 \times 137 \times 5}\left(1 + \frac{1}{\sin(75^\circ)}\right) \cong 93\,\text{fm}$$

which is several orders of magnitude smaller than the radius of the gold atom, thus establishing the nuclear structure of the atom.


Rutherford scattering
2.5
2.0
1.5
1.0
0.5
0.0
y
-2.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
x
asymptote at scattering angle θ = 60°
gold nucleus

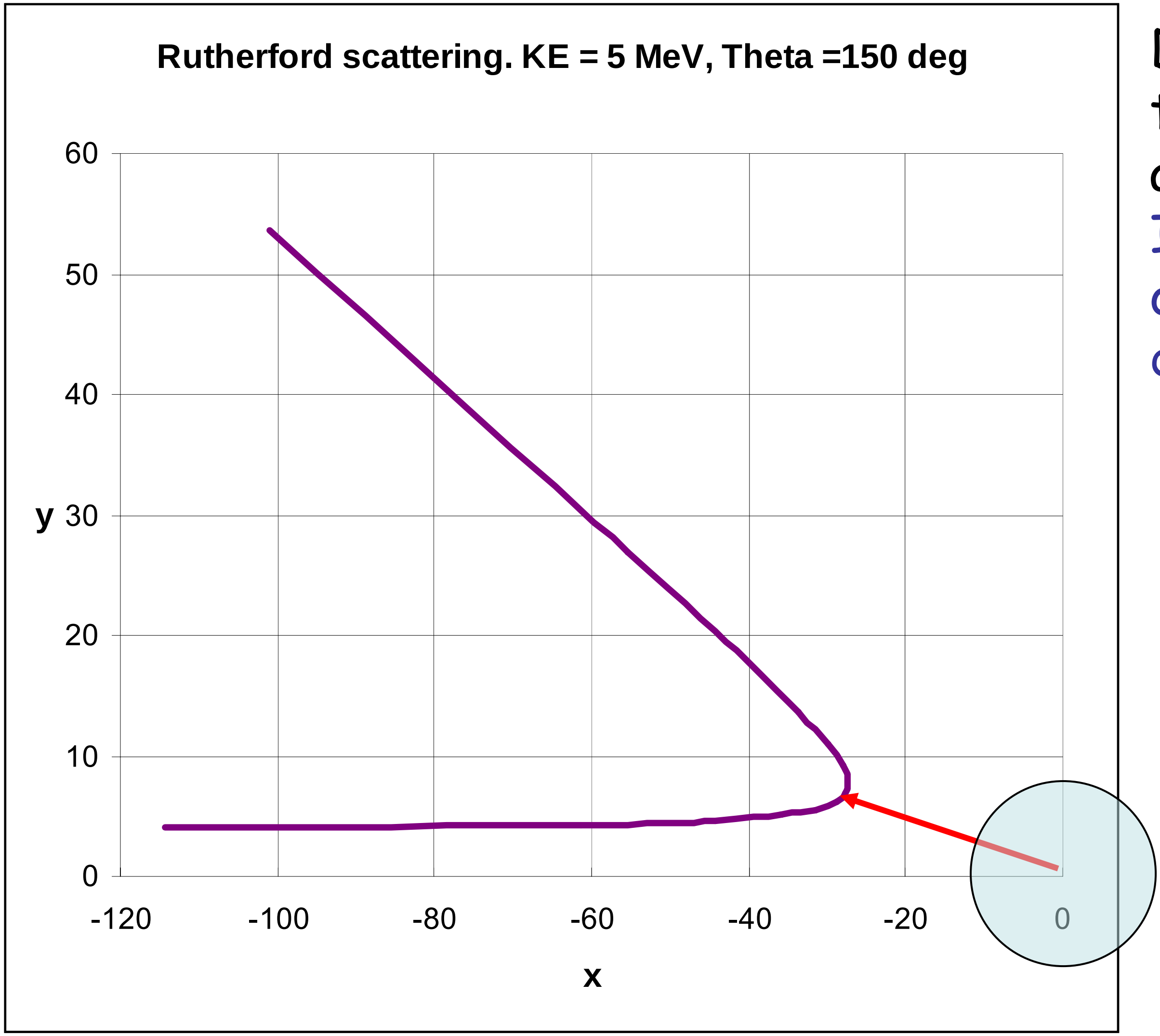


Diagram to illustrate the distance of closest approach.

Ignore the numbers on the axes: the diagram is not to scale.

The result of the Rutherford experiment posed a serious problem.

Indeed, it means that the electrons must be outside the nucleus, and they must be orbiting the nucleus under the attraction of the Coulomb force.

But that implies that the electrons are in an accelerated motion. Therefore, by the laws of electromagnetism, they must continually radiate e.m. waves and hence lose energy. The energy loss must lead to the electrons falling into the nucleus. That is not observed. The continuous emission of e.m. waves should produce a continuous spectrum. That too is not observed.

Therefore one must conclude that the result of the Rutherford experiment is in contradiction to classical electromagnetic theory!
(Recall black-body radiation and the photoelectric effect!)

Niels Bohr proposed a theory of the hydrogen atom that declared that the laws of classical electromagnetism did not apply on the level of atoms.

2. Bohr's Derivation of the Balmer formula.

The derivation assumes the Rutherford nuclear model of the hydrogen atom: the electron of mass m_e and charge $-e$ is orbiting the nucleus (proton) of mass m_p and charge $+e$.

The Coulomb force is

$$\vec{F} = -\frac{e^2}{r^2}\hat{r}$$

This is a central force. Recall Lecture 3 where we have seen that energy and angular momentum are conserved. We write the radius vector as

$$\vec{r} = r\left(\cos\varphi, \sin\varphi, 0\right)$$

and get for the angular momentum

$$L = \mu r^2\dot{\varphi} = \text{constant}, \quad \text{where} \quad \mu = m_e m_p \Big/ \left(m_e + m_p\right)$$

Bohr made the simplifying assumption of circular orbits, hence r = const. and therefore also

$$\dot{\varphi} = \text{constant}, \quad \text{hence} \quad \varphi = \omega t$$

with ω = constant.

Also in Lecture 3 we have introduced the effective potential energy:

$$V_{eff} = \frac{L^2}{2\mu r^2} - \frac{e^2}{r}$$

whose minimum lies at

$$r_0 = L^2 / \mu e^2$$

The total energy of circular motion is equal to the effective P.E. at r_0:

$$E = \min V_{eff} = -\frac{1}{2}\frac{\mu e^4}{L^2} \qquad (1)$$

and we note that this can be written also as

$$E = -\frac{e^2}{2r_0} \qquad (2)$$

So far everything is perfectly classical Newtonian mechanics, but …

now Bohr introduces the ***quantum condition*** in a novel form:

it is not the light quanta as in the Planck-Einstein theory but the ***angular momentum*** that ***is quantized***:

$$L = n\hbar, \quad n = 1, 2, 3, \ldots \tag{3}$$

where $\hbar = h/2\pi$, and substituting this into (1) we get:

$$E_n = -\frac{\mu e^4}{2n^2\hbar^2} \tag{4}$$

It is convenient to replace the charge *e* by the ***fine structure constant*** *α:*

$$\alpha = e^2/\hbar c \tag{5}$$

hence

$$E_n = -\frac{1}{2}\alpha^2 \mu c^2 \cdot \frac{1}{n^2}, \quad n = 1, 2, 3, \ldots \tag{6}$$

Thus, having imposed the quantization condition on the angular momentum, we have got ***quantised energy levels***.

The number n is called ***principal quantum number.***

From (6) we see that the lowest energy level corresponds to $n = 1$.

The state of the atom with $n = 1$ is called the ***ground state***, and E_1 is the ***ground state energy***.

States of the atom with $n > 1$ are ***excited states***: $n = 2$ is the ***first excited state***, $n = 3$ is the ***second excited state***, and so on.

Note also that we get from Eqs. (2) and (6) the radius of the nth orbit:

$$r_n = (n\hbar)^2 / \mu e^2 \qquad (7)$$

The radius of the ground state orbit is called the ***Bohr radius*** of the hydrogen atom.

Normally an atom is in its ground state.

To excite the atom one must supply an ***excitation energy*** equal to the difference between the energy levels.

Atoms can be excited by absorbing e.m. radiation or by collisions with other atoms.

An atom in an excited state is unstable: it will reduce its energy by emitting e.m. radiation or by transferring its excitation energy to another atom in a collision.

The radiative transition from a higher to a lower energy level is, according to Bohr, not a gradual process but a ***quantum jump*** in which a photon of energy

$$E_{mn} = E_m - E_n = \frac{1}{2}\alpha^2 \mu c^2 \left(\frac{1}{n^2} - \frac{1}{m^2} \right)$$

is emitted.

Note the difference with classical e.m. theory!

With the Planck-Einstein relation between energy and frequency of the emitted photon $E = h\nu$ we get

$$\nu_{mn} = \frac{1}{2}\alpha^2 \frac{\mu c^2}{h}\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

The constant

$$R_\mu = \frac{1}{2}\alpha^2 \frac{\mu c^2}{h}$$

is the Rydberg frequency, and

$$R_H = \frac{R_\mu}{c} = \frac{1}{4\pi}\alpha^2 \frac{\mu c^2}{\hbar c}$$

is the Rydberg constant for the hydrogen atom; if we replace the reduced mass by the electron mass we get the ***Rydberg constant***

$$R_\infty = \frac{1}{4\pi}\alpha^2 \frac{m_e c^2}{\hbar c}$$

All constants of nature in the Rydberg constant are known to a high precision:

$$m_e c^2 = 0.510998918(12)\,\mathrm{MeV}$$

$$m_p c^2 = 938.272029(80)\,\mathrm{MeV}$$

$$\hbar c = 197.326968(17)\,\mathrm{MeV\,fm}$$

$$\alpha = 1/137.03599911(46)$$

and hence we can get the Rydberg constant to 7 significant figures:

$$R_\infty = 109737.3\,\mathrm{cm}^{-1}$$

which is also the value found empirically from the measured hydrogen spectrum, after correction from the reduced mass to the electron mass.

To remove the electron from the hydrogen atom in its ground state one must supply an energy of not less than $-E_1$; this is therefore called the ***ionization energy***. Its numerical value is approximately

$$|E_1| \cong 13.6\,\mathrm{eV}$$

a value whose order of magnitude it is useful to remember for processes on the atomic scale.

From this introduction to quantum physics it may appear that the only difference between classical and quantum physics is in a modification needed of electrodynamics, and that classical mechanics needs no fundamental revision.

Indeed, Bohr's quantum condition imposed on the angular momentum of the hydrogen atom is hardly a deep change of classical mechanics, as it leaves Newton's three laws of mechanics unchanged.

But this impression would be wrong. About ten years after Bohr's work it was realised that the ***wave-particle duality*** applies to all matter, and not only to electromagnetic radiation.

The theory that takes account of the wave property of matter is ***wave mechanics***. But wave mechanics goes beyond the level of the present course.

In this course we shall continue with a review of the basic phenomena of ***nuclear physics***.