Selected Topics in Physics

a lecture course for 1st year students

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Lecture 11

1.) Determination of parameters of the SEMF

2.) α decay

3.) Nuclear energy levels

4.) Spontaneous fission

1.) Determination of parameters of the SEMF

1.1) Determination of the Coulomb term from mirror nuclei
1.2) Determination of the asymmetry term from line of maximum β stability
1.3) Estimate of the pairing term

1.1) Determination of the Coulomb term from mirror nuclei

In Lecture 9 we have derived a formula for the binding energy, based on the liquid drop model:

$$B(A,Z) = a_1 A - a_2 A^{2/3} - a_3 \frac{Z(Z-1)}{A^{1/3}} - a_4 \frac{(N-Z)^2}{A} + \delta$$

where the terms are, in that order: ***volume*** term, ***surface*** term, ***Coulomb*** term, ***asymmetry*** term and ***pairing*** term.

We can see that the volume and surface terms are identical for isobars. Therefore, if we compare the binding energies of two isobars, then these terms cancel.

If we choose these isobars to be odd-A mirror nuclides, then the asymmetry term also cancels.

Mirror nuclei:

Two nuclides are called ***mirror nuclei*** if they have equal mass numbers A, and if the number of protons Z in one of them is equal to the number of neutrons N in the other.

Examples of mirror nuclei:

$$\text{boron-11 and carbon-11} \quad : \quad {}^{11}_{5}B - {}^{11}_{6}C$$

$$\text{carbon-13 and nitrogen-13} : \quad {}^{13}_{6}C - {}^{13}_{7}N$$

If we define the ***neutron excess*** of a nucleus by

$$T = N - Z$$

(also called *isotopic number*), then we have for odd-A mirror nuclei

$$T = \pm 1, \quad \text{and hence} \quad Z = N \pm 1$$

Thus these mirror nuclides lie next to the line $N = Z$ in the Segre chart.

Now we know from the Segre chart that the band of stability moves with increasing A away from the line $N = Z$ towards neutron-rich nuclides, and unstable nuclides are never very far away from the band of stability.

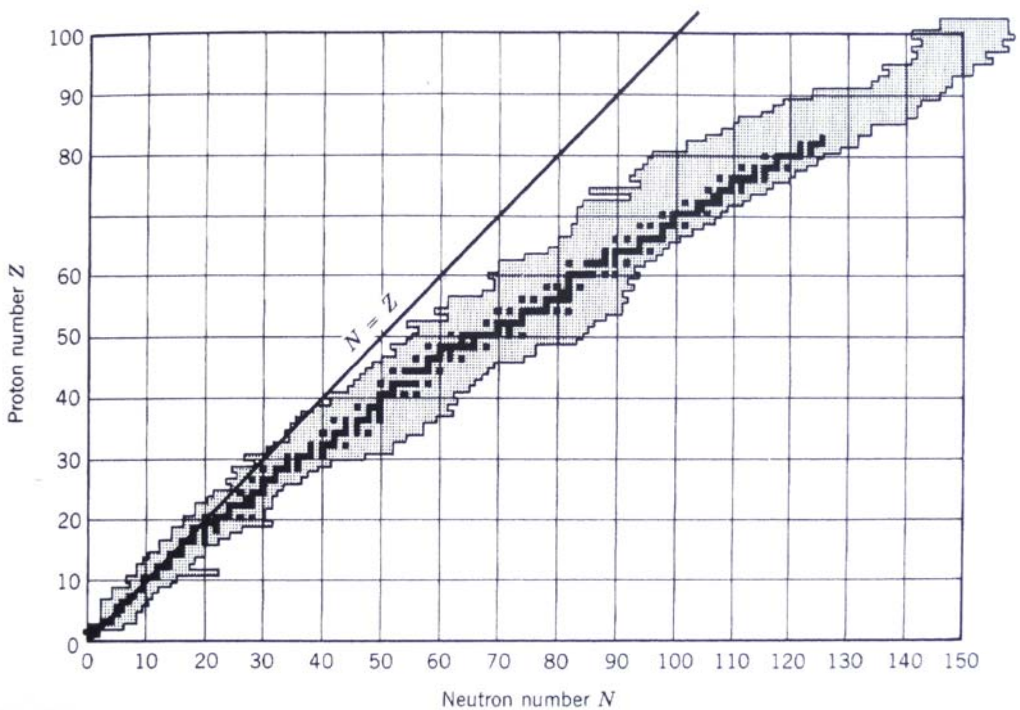


Figure 1.1 Stable nuclei are shown in dark shading and known radioactive nuclei are in light shading.

Therefore we must expect that mirror nuclides are found only up to mass numbers A around 60.

All mirror nuclides with A >10 are shown in the following table

Mirror nuclides from A = 11 to A = 63: there are no mirror nuclides with A > 63

	N - Z = +1			N - Z = -1		
A	Nuclide	Z	Delta	Nuclide	Z	Delta
11	B	5	8.668	C	6	10.650
13	C	6	3.125	N	7	5.345
15	N	7	0.101	O	8	2.856
17	O	8	-0.809	F	9	1.952
19	F	9	-1.487	Ne	10	1.751
21	Ne	10	-5.732	Na	11	-2.184
23	Na	11	-9.530	Mg	12	-5.474
25	Mg	12	-13.193	Al	13	-8.916
27	Al	13	-17.197	Si	14	-12.384
29	Si	14	-21.895	P	15	-16.953
31	P	15	-24.441	S	16	-19.045
33	S	16	-26.586	Cl	17	-21.003
35	Cl	17	-29.014	Ar	18	-23.047
37	Ar	18	-30.948	K	19	-24.800

	N - Z = +1			N - Z = -1		
A	Nuclide	Z	Delta	Nuclide	Z	Delta
39	K	19	-33.807	Ca	20	-27.274
41	Ca	20	-35.135	Sc	21	-28.642
43	Sc	21	-36.188	Ti	22	-29.321
45	Ti	22	-39.006	V	23	-31.880
47	V	23	-42.002	Cr	24	-34.560
49	Cr	24	-45.331	Mn	25	-37.620
51	Mn	25	-48.241	Fe	26	-40.220
53	Fe	26	-50.945	Co	27	-42.640
55	Co	27	-54.028	Ni	28	-45.340
57	Ni	28	-56.082	Cu	29	-47.310
59	Cu	29	-56.357	Zn	30	-47.260
61	Zn	30	-56.350	Ga	31	-47.090
63	Ga	31	-56.547	Ge	32	-46.900

Delta is the mass excess in MeV;

the values are from the BNL Wallet Cards 2005, **http://www.nndc.bnl.gov/wallet/**

Let us now compare the binding energies of mirror nuclides using the SEMF:

we have

$$B(A,Z) = a_1 A - a_2 A^{2/3} - a_3 \frac{Z(Z-1)}{A^{1/3}} - a_4 \frac{(N-Z)^2}{A} + \delta$$

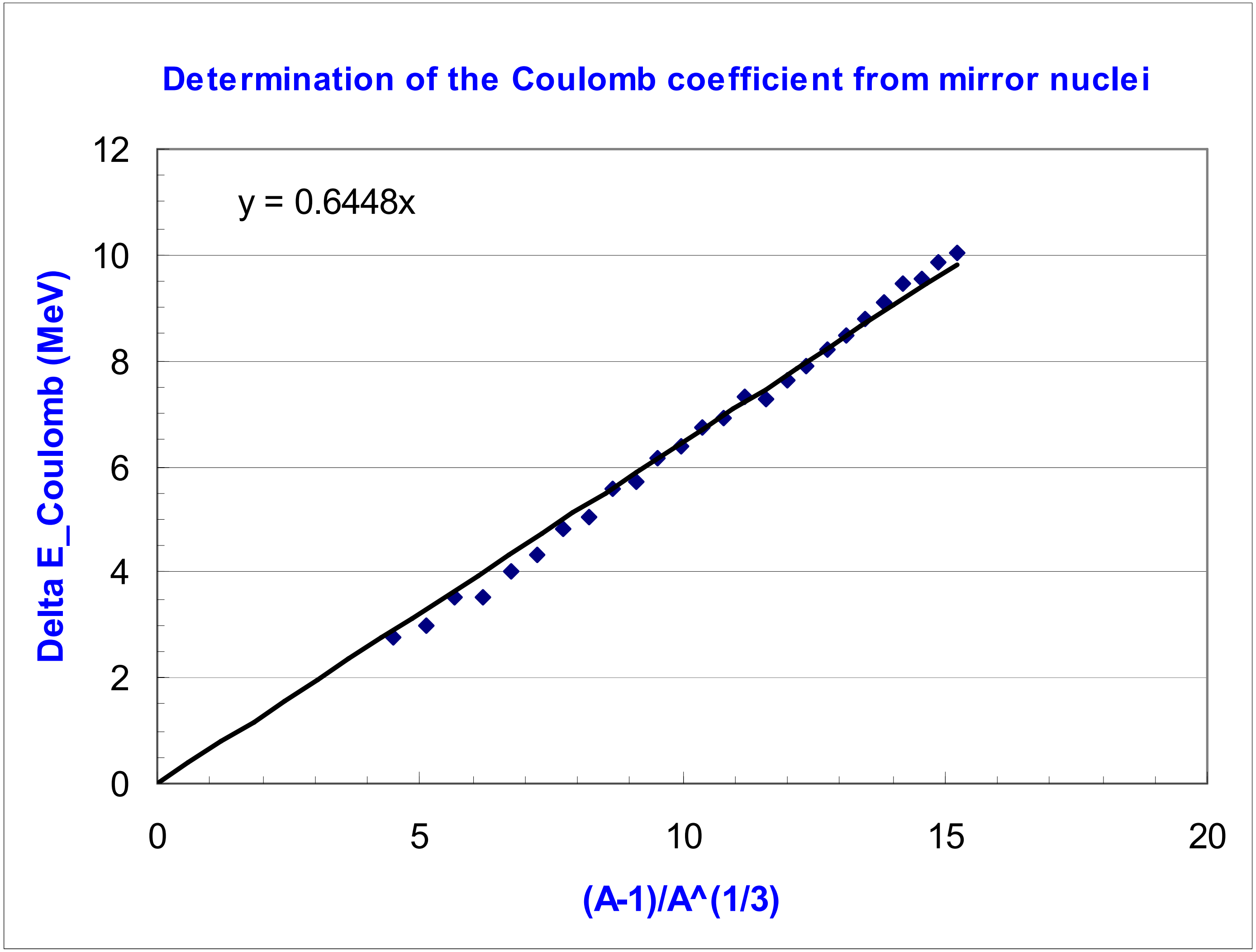
or in terms of A and $T = N - Z$:

$$B(A,T) = a_1 A - a_2 A^{2/3} - \frac{1}{4} a_3 \frac{(A-T)(A-T-2)}{A^{1/3}} - a_4 \frac{T^2}{A} + \delta$$

hence

$$\Delta B \equiv B(A,T=+1) - B(A,T=-1)$$
$$= -\frac{1}{4} a_3 \frac{(A-1)(A-3)-(A+1)(A-1)}{A^{1/3}}$$
$$= a_3 (A-1) A^{-1/3}$$

Thus, if we plot the measured differences of the binding energies as a function of $(A-1)/A^{1/3}$, then we expect a straight line through the origin. This is shown in the next figure.



The slope of the line, which is constrained to pass through the origin, is the empirical coefficient a_3 of the ***Coulomb*** term. The least squares fit gives

$$a_3 = 0.645\,\text{MeV}$$

1.2) Determination of the asymmetry term from line of maximum β stability

In the previous lecture we have defined the position of maximum β stability:

$$\frac{\partial M(A,Z)}{\partial Z}\bigg|_{A=const} = 0$$

Now, the mass $M(A,Z)$ is given by

$$M(A,Z) = Zm_H + (A-Z)m_n - B(A,Z)/c^2$$

hence

$$Z_{\min} = \frac{m_n c^2 - m_H c^2 + a_3 A^{-1/3} + 4a_4}{2a_3 A^{-1/3} + 8a_4 A^{-1}}$$

thus the position of the minimum of the mass parabola is determined by the coefficients a_3 (Coulomb term) and a_4 (asymmetry term).

Solving for a_4 we get

$$a_4 = \frac{m_n c^2 - m_H c^2 + a_3 \left(1 - 2Z_{\min}\right) A^{-1/3}}{8Z_{\min} A^{-1} - 4}$$

Empirically, fitting the masses of isobars with parabolas, we will get somewhat different values of a_4. Then these values are averaged.

From fits to only a few mass parabolas of mass numbers A = 15, 63, 65, 97, 101, 107 and 135 I have got the following result:

$$a_4 = 21.8 \pm 1.3 \,\mathrm{MeV}$$

1.3) Estimate of the pairing term

The pairing term δ is nonzero only for even-A nuclides.

In lecture 9 we have seen in the discussion of stability rules that of the 174 stable even-A nuclides, 166 are even-even and only 8 are odd-odd nuclides.

Four of the odd-odd stable nuclides are light nuclides:
deuterium, lithium, boron and nitrogen;
they are absolutely stable and symmetric, i.e. $N = Z$.

The remaining four odd-odd nuclides are unstable but have lifetimes greater then 10^9 years, *i.e.* greater than the age of the earth:

${}^{40}_{19}K$	1.3×10^{9} *years*
${}^{50}_{23}V$	1.4×10^{17} *years*
${}^{138}_{57}La$	1.1×10^{11} *years*
${}^{180m}_{73}Ta$	$>1.2\times10^{15}$ *years*

We will be guided by these stability rules to estimate the pairing term δ

To do this we rewrite the expression for the minimum of the mass parabola in terms of the neutron excess T:

$$T_{\min} = A - 2Z_{\min} = \frac{a_3(A-1)A^{-1/3} - (m_n - m_H)}{a_3 A^{-1/3} + 4a_4 A^{-1}}$$

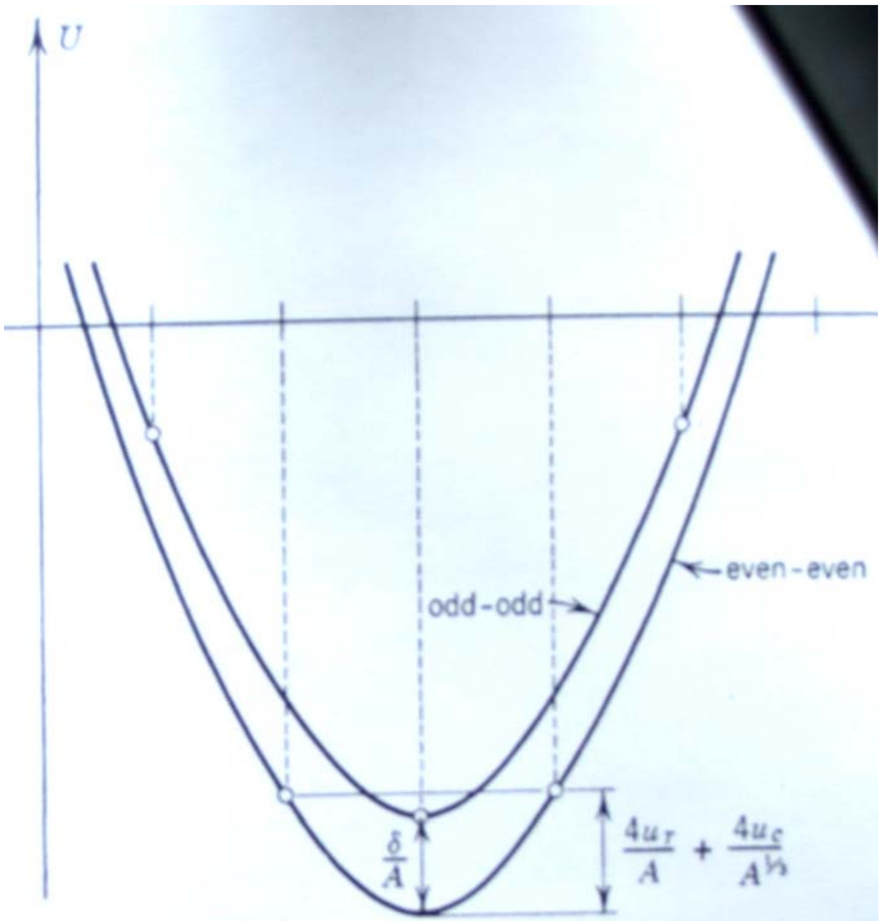
and then use this to get the following result:

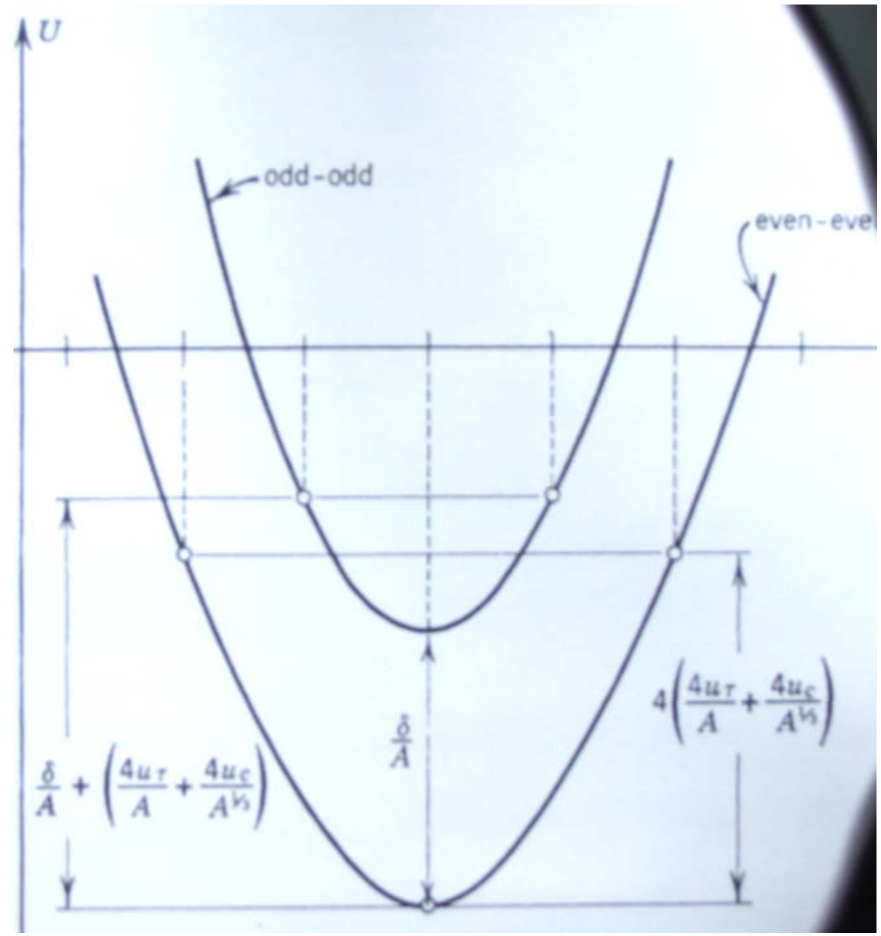
$$\Delta U \equiv M(A,T)c^2 - M(A,T_{\min})c^2$$
$$= \left(\frac{1}{4}a_3 A^{-1/3} + a_4 A^{-1}\right)(T - T_{\min})^2$$

Then we apply this to even-A nuclides. The stability rules suggest that the mass parabolas of the odd-odd nuclides are shifted upwards from the mass parabolas of the even-even nuclides. We denote the separation between these parabolas by δ.

Mass parabolas for even-A nuclides: the energy U=Mc^2 is plotted vs T

(Figures from Blatt and Weisskopf; their notation differs from ours: they write δ/A instead of our δ, also $4u_C = a_3$ and $4u_T = a_4$)





Shown on the left is the situation where there is one stable odd-odd nuclide at $T = T_{\text{min}}$ and no stable even-even nuclides. Here

$$\delta < \left(\frac{1}{4} a_3 A^{-1/3} + a_4 A^{-1} \right)$$

From the stability rules we know that this does not happen in nature. Therefore we conclude that

$$\boxed{\delta > \left(\frac{1}{4} a_3 A^{-1/3} + a_4 A^{-1} \right)}$$

On the right-hand figure there are three stable even-even nuclides, the middle one at $T = T_{\text{min}}$. Their stable even-even neighbors are separated by 2 units in T, and hence their energy difference is

$$\delta U = 4 \left(\frac{1}{4} a_3 A^{-1/3} + a_4 A^{-1} \right)$$

The unstable odd-odd nuclides lie above the minimum of their mass parabola by ΔU with T - T_{min} = 1, and therefore above the minimum of the mass parabola of the even-even nuclides by $\delta + \Delta U$, thus

$$\delta + \left(\frac{1}{4} a_3 A^{-1/3} + a_4 A^{-1} \right) > 4 \left(\frac{1}{4} a_3 A^{-1/3} + a_4 A^{-1} \right)$$

hence

$$\delta > 3 \left(\frac{1}{4} a_3 A^{-1/3} + a_4 A^{-1} \right)$$

This situation occurs rarely in nature; we must therefore conclude, together with the previous result, that

$$\frac{1}{4} a_3 A^{-1/3} + a_4 A^{-1} < \delta \leq 3 \left(\frac{1}{4} a_3 A^{-1/3} + a_4 A^{-1} \right)$$

Now, we had $\delta = a_5/A$, and that gives us an estimate of a_5:

$$\boxed{\frac{1}{4} a_3 A^{2/3} + a_4 < a_5 \leq 3 \left(\frac{1}{4} a_3 A^{2/3} + a_4 \right)}$$

For a numerical estimate let us take an intermediate value of A = 100; then, with our values for a_3 and a_4 we get

$$32 < a_5 \leq 96$$

but for small values of A the limits get tighter: if we put A = 1, then

$$22 < a_5 \leq 65$$

We must conclude that this parameter can be determined only semi-quantitatively. But then we must remember that the basis of the SEMF, the liquid drop model of nuclear matter, is only the simplest model that gives at least reasonable results. More sophisticated models require quantum mechanics. These are outside the scope of the present lecture course.

We will continue using the SEMF next to discuss the (*in*)stability of nuclei against α decay.

2.) α decay

The reaction equation of α decay is

$$ {}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}\alpha + Q $$

where Q is the kinetic energy released in the process.

In the rest frame of the mother nucleus X the momenta of the daughter nucleus Y and the α particle are equal in magnitude.

From the previous lecture we have the following formula for the momentum

$$ p^{*} = \frac{1}{2M_X}\left\{\left[M_X - \left(M_Y - m_\alpha\right)^2\right]\left[M_X - \left(M_Y + m_\alpha\right)^2\right]\right\}^{1/2} $$

Thus for example in the α decay of radium-226

$$ {}^{226}_{88}Ra \rightarrow {}^{222}_{86}Rn + {}^{4}_{2}\alpha + Q $$

we find p^* = 189 MeV/c, and hence kinetic energies …

of the radon nucleus and the α particle:

$$KE_\alpha = \sqrt{p^{*2} + m_\alpha^2} - m_\alpha = 4.78\ \mathrm{MeV},$$

$$KE_{Rn} = \sqrt{p^{*2} + M_{Rn}^2} - M_{Rn} = 0.09\ \mathrm{MeV}$$

as a general rule the kinetic energy of the daughter nucleus is small compared with the kinetic energy of the α particle.

For the decay to take place the Q value must be greater than zero

Thus the condition for α decay is

$$Q = M\left(A, Z\right)c^2 - M\left(A-4, Z-2\right)c^2 - m\left({}^4He\right)c^2 > 0$$

or in terms of the mass excess $\Delta = M(A,Z) - A$

$$\Delta\left(A, Z\right) - \Delta\left(A-4, Z-2\right) - \Delta\left({}^4He\right) > 0$$

and hence, taking the values of Δ from tables of nuclides, …

we have

$$\Delta\left({}^{226}Ra\right) = 23.669 \text{ MeV},$$

$$\Delta\left({}^{222}Rn\right) = 16.374 \text{ MeV},$$

$$\Delta\left({}^{4}He\right) = 2.425 \text{ MeV}$$

and hence

$$Q\left({}^{226}Ra \rightarrow {}^{222}Rn + \alpha\right) = 4.78 \text{ MeV}$$

Empirically one knows that spontaneous α decay of naturally occurring nuclides takes place only for ***heavy*** nuclides. The lightest α unstable nuclides are shown in the next table

A	Z	Element	$T_{1/2}$
144	60	Nd	2.3E18
147	62	Sm	1.06E11
148	62	Sm	7E18

We can see that these nuclides are barely unstable: they much prefer not to decay.

Now let us see whether the SEMF is qualitatively in agreement with the empirical evidence.

From the SEMF we find the following expression for the Q value:

$$Q = M(A,Z)c^2 - M(A-4,Z-2)c^2 - m\left({}^4He\right)c^2$$
$$= B\left({}^4He\right) - 4a_1 + \frac{8}{3}\frac{a_2}{A^{1/3}} + 4a_3\frac{Z}{A^{1/3}}\left(1-\frac{Z}{3A}\right) - 4a_4\left(1-2\frac{Z}{A}\right)^2$$

and if we want to apply the condition to naturally occurring nuclides, then we must put Z equal to its value on the line of β stability:

$$Z \simeq A\left(2 + a_3 A^{2/3} / 2a_4\right)$$

This is not the kind of formula that allows us to see at a glance what's going on, so we better write a little computer program in our preferred language (which in my case is FORTRAN) to compute Q as a function of A. The results will depend somewhat on the values of the constants we use. With the values of Lecture 9 we get the following result:

$$\begin{array}{lll} Q < 0 & for & A \leq 146 \\ Q > 0 & for & A \geq 147 \end{array}$$

which is in better than just qualitative agreement with the empirical evidence.

3.) Nuclear energy levels

Since nuclei consist of nucleons we may expect by analogy with atoms that there can be excited nuclear states.

To make a transition from its ground state into an excited state the nucleus must absorb energy.

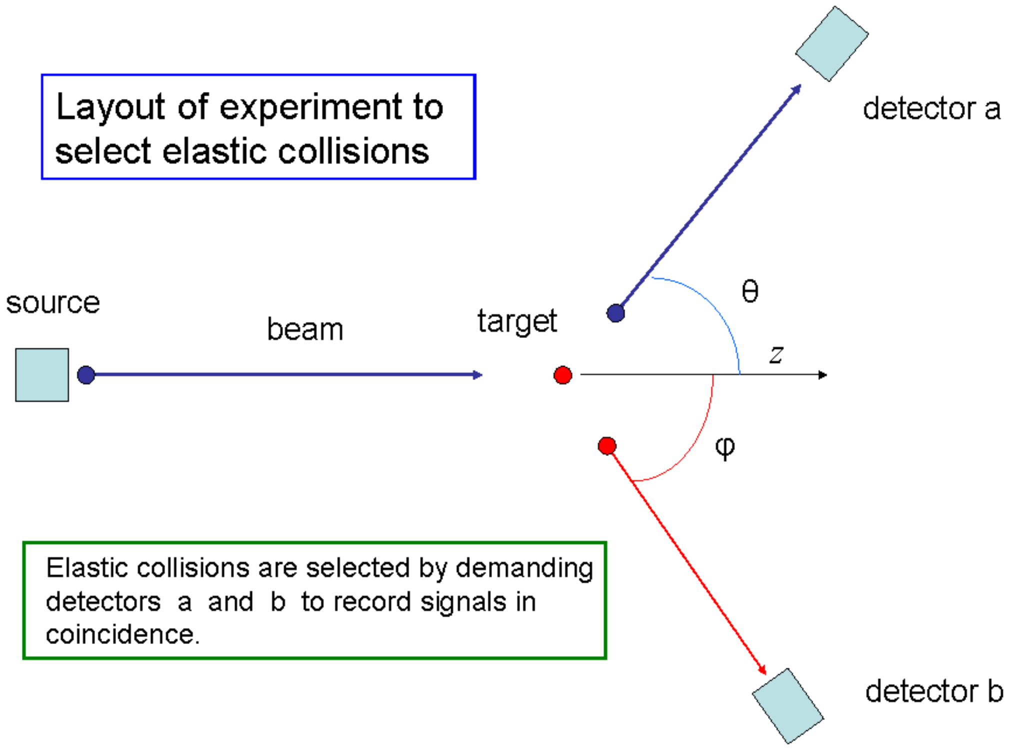
Excitation energy can be transferred to a nucleus by a collision with another nucleus or by exposure to electromagnetic radiation.

An experiment to demonstrate the existence of excited nuclear states is by collisions with protons.

A beam of protons is directed at a target of some pure substance. Measured are the energies of the scattered protons at some fixed scattering angle θ

This experiment is a slight modification of the elastic scattering experiment discussed in a previous lecture, so I did not produce a new figure.

We denote the 4-momenta of the beam particle by **p** and of the target particle by **P**; the 4-momenta of the scattered and recoil particles are **p'** and **P'**.



By 4-momentum conservation we have

$$p + P = p' + P'$$

hence

$$P'^2 = \left(p + P - p'\right)^2$$

Let us denote the masses of the incident particle, target particle, scattered particle and struck particle by

$$m_p, \quad M_t, \quad m_p, \quad M$$

hence

$$M^2 = M_t^2 + 2M_t\left(E - E'\right) + 2m_p^2 - 2\vec{p}\cdot\vec{p}'$$

Observe the scattered protons at 90 degrees to the beam direction, then

$$\vec{p}\cdot\vec{p}' = 0$$

and if we express the energy in terms of the kinetic energy:

$$E = T + m_p$$

then

$$T' = \frac{1}{2}\frac{M_t^2 - M^2 + 2T\left(M_t - m_p\right)}{M_t + T + m_p}$$

In the particular case of elastic scattering the formula simplifies:

$$T'_{elastic} = T\,\frac{M_t - m_p}{M_t + T + m_p}$$

and then we can write the formula for the general case as

$$T' = T'_{elastic} - \frac{1}{2}\frac{M^2 - M_t^2}{M_t + T + m_p}$$

Thus we expect to see protons with kinetic energies less than the kinetic energy of elastically scattered protons.

In a particular experiment, protons of kinetic energy 10.02 MeV were incident on a target containing boron-10 whose atomic mass is 10.01 u. Thus, remembering that 1 u = 931.494 MeV, we get

$$T'_{elastic} = 8.18\ \mathrm{MeV}$$

In the figure the proton KE is plotted along the horizontal axis, and vertically the number of protons in arbitrary units. The discrete set of lines is a clear demonstration of nuclear energy levels.

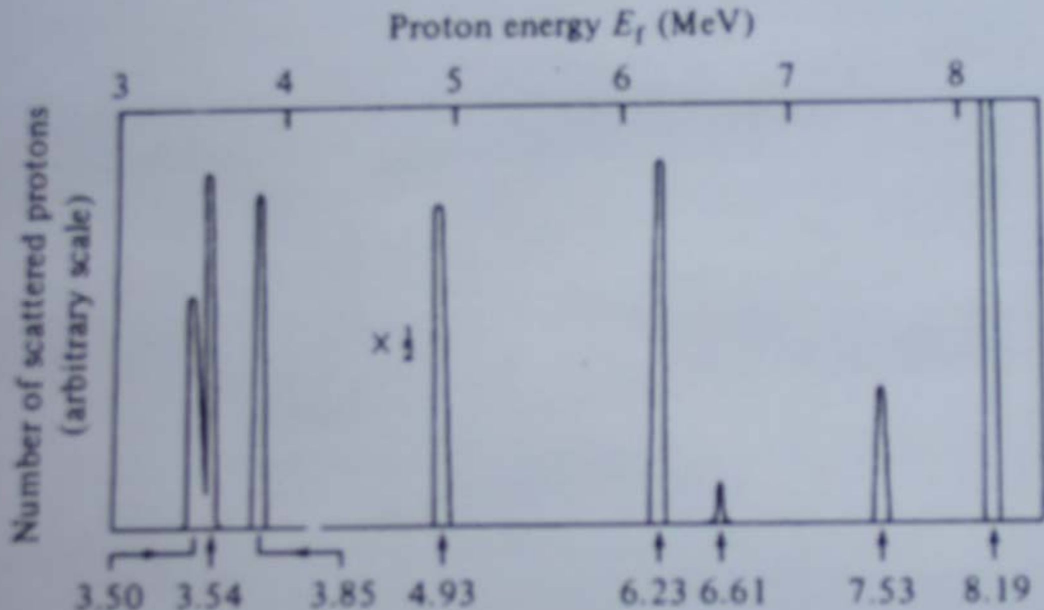
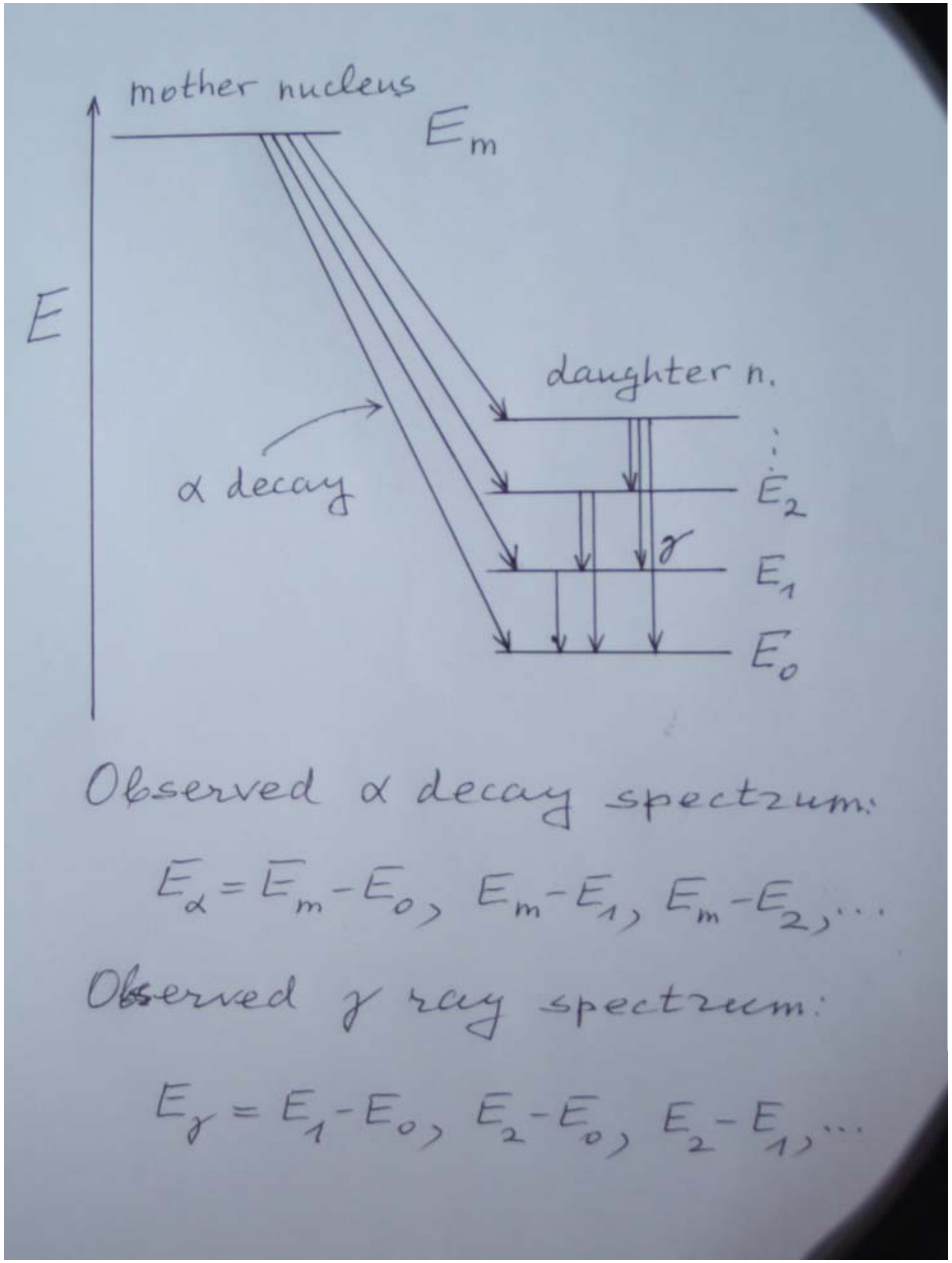


Fig. 7.2 The number of protons scattered at 90° from a static target containing ^{10}B, as a function of their final energy E_f. Initially the protons were in a collimated beam and had energy 10.02 MeV. Background scattering has been removed. (Data from Armitage, B. H. & Meads, R. E. (1962), *Nucl. Phys.* **33**, 494.)

Another way of producing excitation is by decay of a nucleus. Then it can happen that the daughter nucleus is in an excited state.

In the figure this is shown for the example of α decays into the different energy levels of the daughter nucleus.

Observed is a discrete set of α energies and associated emission of γ rays



4.) Spontaneous fission

Nuclear fission is the process of separation of a nucleus into fragments of similar size; this is usually accompanied by emission of one or several neutrons.

Fission can take place spontaneously or by induction. The usual mechanism of induced fission is the bombardment of the nucleus with neutrons.

Consider the spontaneous fission of nucleus X into fragments A and B:

$$X \to A + B + Q$$

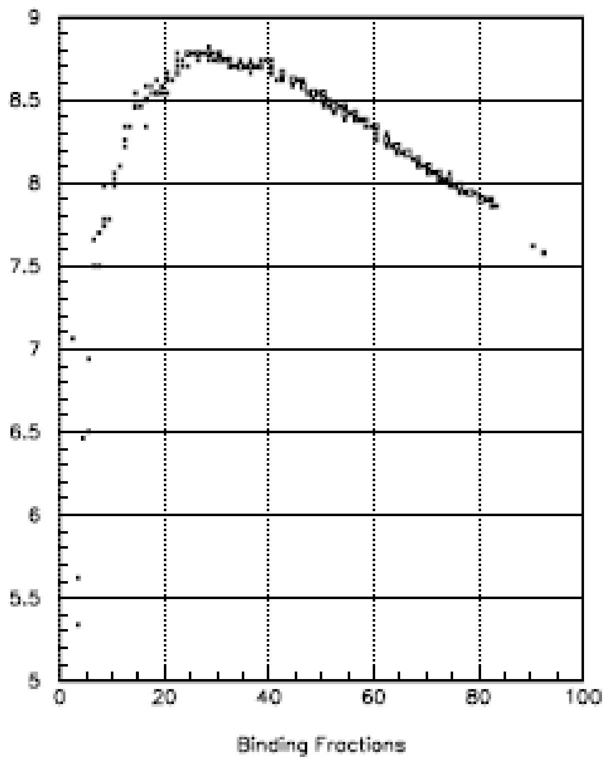
The masses of the fragments are usually about equal. Let us for simplicity put

$$m(A) = m(B) = \frac{1}{2} m(X)$$

then the released kinetic energy is

$$Q = B(A) + B(B) - B(X) = 2B(A) - B(X)$$

Now, the binding fraction at A = 240 is about 7.6 MeV per nucleon, and at A = 120 about 8.5 MeV per nucleon. Therefore



$$Q \simeq (8.5 - 7.6)A \sim 200\,\mathrm{MeV}$$

The fragments have an excess of neutrons. This can be seen by reference to the Segre chart

By dividing the heavy nucleus into two fragments of about equal mass, the daughter nuclei end up further from the line of stability into the region of greater neutron excess.

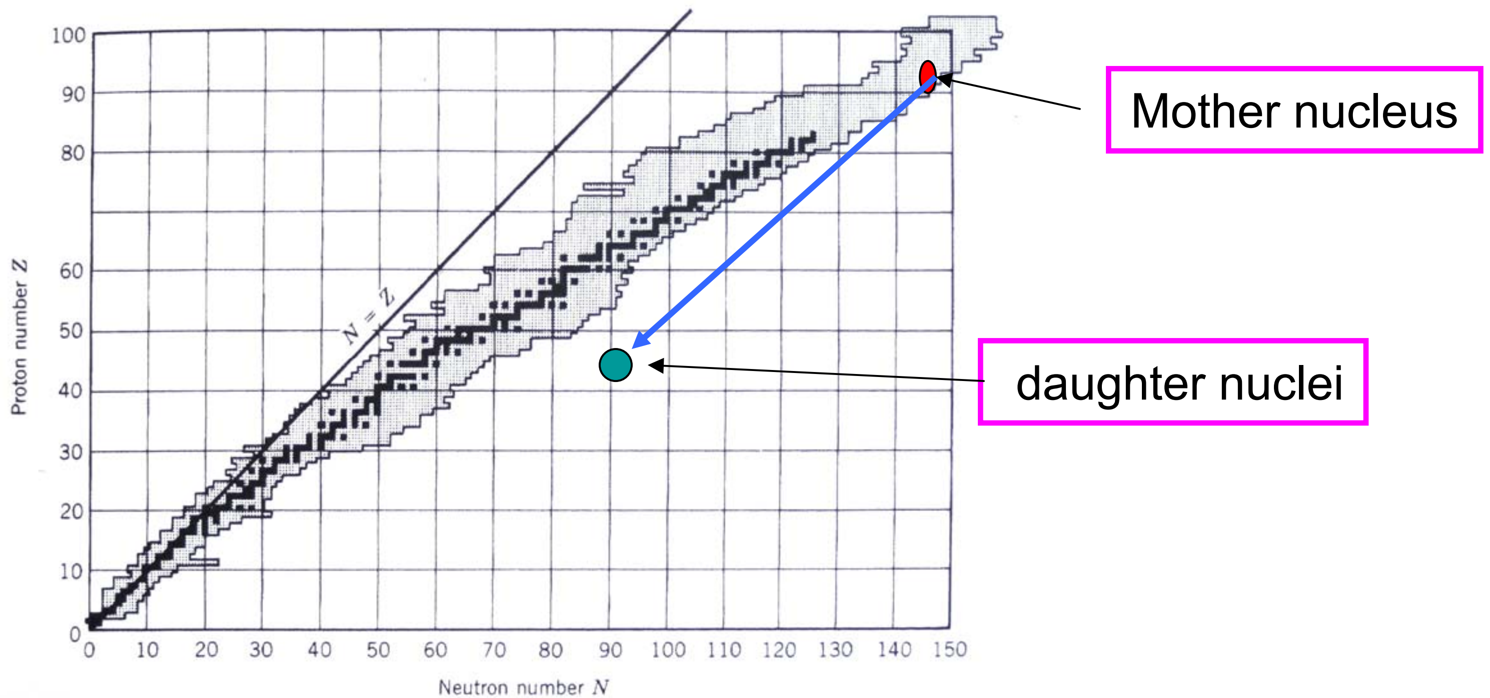


Figure 1.1 Stable nuclei are shown in dark shading and known radioactive nuclei are in light shading.

The excess of neutrons can be reduced by emission of neutrons and/or by β decay. Usually both processes take place.

The fragments are initially in excited states; de-excitation takes place by emission of gamma rays.

Some of the kinetic energy is carried by the neutrinos which are always emitted together with electrons in beta decay. These neutrinos escape without depositing their energy in the surrounding material. All other particles are absorbed. The absorbed energy is ultimately converted into heat.

To produce 1 Joule of thermal energy requires of the order of 10^{10} fissions

$$1 \text{ Joule} = 0.623\times10^{13} \text{ MeV}$$

and we get about 200 MeV per fission:

$$200 \text{ MeV}/\,fission = \frac{200}{0.623} 10^{-13} \text{ Joule}/\,fission = \frac{1 \text{ Joule}}{N \;\; fissions}$$

hence

$$N = \frac{0.623}{200} 10^{13} \simeq 3 \times 10^{10} \; fissions \;\; per \; \text{Joule}$$

Bohr-Wheeler Theory of Nuclear Fission

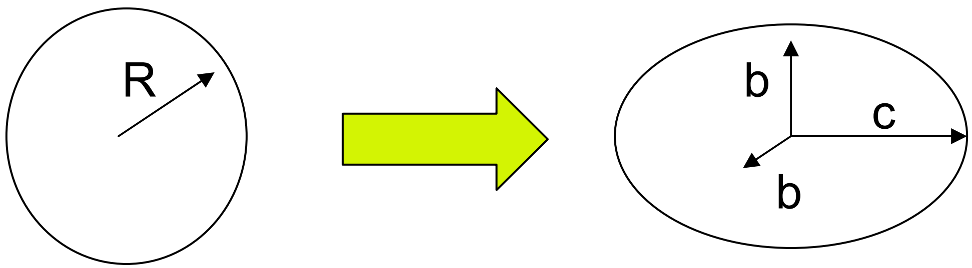
In the liquid drop model, a deformation of the nucleus takes place without change of the volume.

As a result of the deformation the energy of the nucleus changes. The energy difference between the deformed and undeformed nuclei is the ***deformation energy***.

Fission can occur if the deformation energy is less than zero.

Estimate of the deformation energy: in the SEMF only the surface energy and the Coulomb energy change.

Consider a small deformation of a sphere into an ellipsoid:



$$V = \frac{4\pi}{3} R^3 \longrightarrow V = \frac{4\pi}{3} b^2 c$$

Let

$$c = R(1+\varepsilon) \quad \text{hence} \quad b = R \Big/ \sqrt{(1+\varepsilon)}$$

(ε = deformation parameter, |ε| << 1)

Surface energy:

$$a_2 A^{2/3} \rightarrow a_2 A^{2/3} \left(1 + \frac{2}{5}\varepsilon^2\right)$$

Coulomb energy:

$$a_3 Z^2 A^{-1/3} \rightarrow a_3 \left(Z^2 / A^{1/3}\right)\left(1 - \frac{1}{5}\varepsilon^2\right)$$

(**Exercise!!!**)

hence net ***increase*** of E as a result of the deformation (***deformation energy***):

$$\Delta E = \frac{1}{5}\varepsilon^2 \left(2a_2 A^{2/3} - a_3 Z^2 / A^{1/3}\right) = k\varepsilon^2 \quad (\text{definition of } k)$$

i.e.

$$k = \frac{1}{5}\left(2a_2 A^{2/3} - a_3 Z^2 / A^{1/3}\right)$$

Condition for stability: $k > 0$

Condition for ***in***stability $k < 0$

Critical value: $k = 0$

hence

$$2a_2 A^{2/3} - a_3 Z^2 / A^{1/3} = 0$$

or

$$Z^2 / A = 2a_2 / a_3$$

and with

$$a_2 = 13\,\text{MeV}, \quad a_3 = 0.6\,\text{MeV}$$

we get

$$Z^2 / A \simeq 43.5$$

and hence with Z on the line of maximum β stability we get

$$k \le 0 \quad for \quad Z \ge 110$$