Selected Topics in Physics

a lecture course for 1st year students

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1. Classical Mechanics
2. Special Relativity
3. The Rutherford-Bohr Model of the Atom
4. Nuclear and Elementary Particle Physics

1. Classical Mechanics

1.1 Space and time in Newtonian mechanics; velocity, acceleration, mass, momentum, force. Newton's three laws of motion.

1.2 **Examples:** projectile in constant gravitational field and in $1/r^2$ field.

1.3 **Oscillations:** simple harmonic oscillations; coupled oscillations, beats; damped oscillations.

1.4 **Energy:** kinetic and potential energy, conservation of total energy.

1.5 Angular momentum.

1.6 The solar system; tidal force.

… The clouds of ignorance
At last dispelled by science. …

Arise! And, casting off your earthly cares,
Learn ye the potency of heaven-born mind,
Its thought and life far from the herd withdrawn!

From the ***Ode Dedicated to Newton***
by Edmond Halley (1656-1742),
Astronomer Royal

… rational mechanics will be the science of motions resulting from any forces whatsoever, and of the forces required to produce any motions, accurately proposed and demonstrated.

from Newton's Preface to the First Edition of his ***Mathematical Principles of Natural Philosophy***, Cambridge, 1686.

1.1 Space and time in Newtonian mechanics; velocity, acceleration, mass, momentum, force. Newton's three laws of motion.

In physics, the definition of **SPACE** is based on the need to specify quantitatively the positions of material objects. This can be done only in relation to other material objects.

For many purposes the top of an experimental table in the laboratory is sufficient: we can run coordinate axes along the edges of the table and describe the position of some object on the table in terms of the distances from the axes.

But the laboratory is attached to the surface of the earth and is involved in a complicated movement: the earth rotates about its axis, revolves about the sun and together with the sun revolves about the centre of our galaxy. Therefore, for some purposes we use coordinate frames centred on the sun with axes pointing to distant stars: these are solar coordinates. Similarly we can define galactic coordinates.

The physical concept of **TIME** also requires the presence of matter. Moreover, the matter must be involved in ***processes***, changing its position or its composition.

A simple process that allows us to distinguish between the past and the future is the breaking of a stick of chalk: everything that was before the stick of chalk broke is the past, and at the instant the chalk breaks the future begins.

To ***measure*** time we use ***periodic*** processes. We use the rotation of the earth about its axis that appears to us as periodic movements of the sun, moon and stars.

But the earth is not a perfect sphere, and its interior is in constant motion, visible by the drift of the continents. Therefore the time, measured by the apparent motion of celestial objects, is also not perfect.

We have good reason to believe that oscillations of atoms are more regular and we use atomic clocks for the most precise measurement of time.

Once we have a concept of space and time that can be expressed in numbers, we can describe the ***changing positions of objects***.

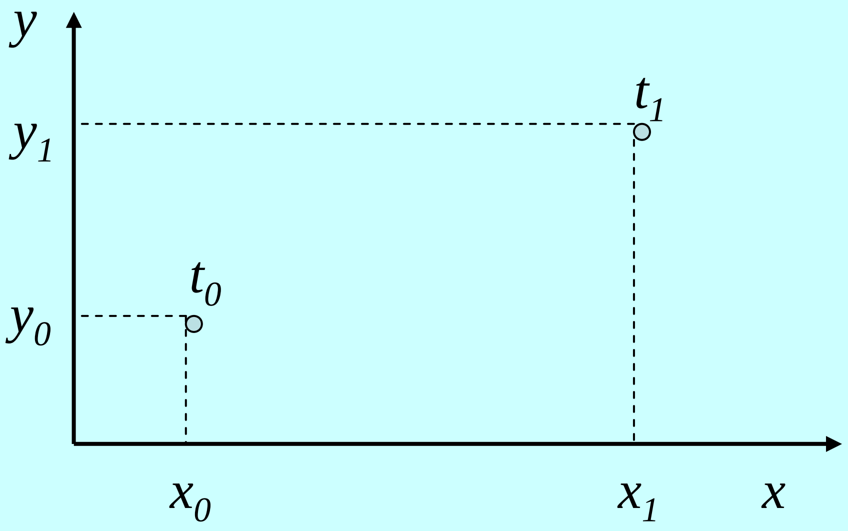
For many purposes in classical mechanics a convenient object is a material point which I shall call ***particle***.

If a particle is at a certain time t_0 at a position, described in terms of ***Cartesian coordinates*** (x_0, y_0, z_0), and at a later time t_1 at (x_1, y_1, z_1), then we say that its ***average velocity*** during the interval of time $\Delta t = t_1 - t_0$ was

$$\vec{v}_{av} = \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t} \right)$$

where

$$\Delta x = x_1 - x_0,\ \Delta y = y_1 - y_0,\ \text{and}\ \Delta z = z_1 - z_0.$$



Definition of average velocity (z coordinate not shown):

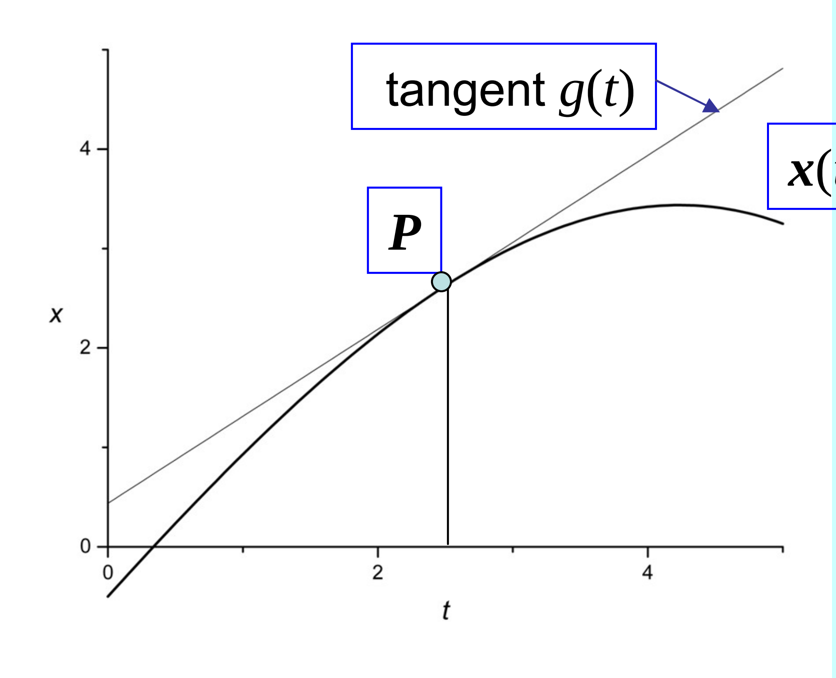
$$\vec{v}_{av} = \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t} \right)$$

In the limit of $\Delta t \longrightarrow 0$, we get the ***instantaneous velocity***

$$\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$

where

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \quad \frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}, \quad \frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}$$



Definition of instantaneous velocity (only one spatial dimension shown):

the x-component of the velocity is defined by

$$v_x(t) = \frac{dx(t)}{dt}$$

it is equal to the slope of the tangent $g(t)$ to the curve $x=x(t)$ at point P

The y and z components of velocity are defined similarly.

The x, y, and z components of the velocity are combined into the ***velocity vector***:

$$\vec{v} = \left(v_x, v_y, v_z\right)$$

Everyday experience tells us that the velocities of objects also change with time. Similarly to the velocity, which we have defined as the *rate of change of the position of the object*, we define the ***acceleration*** as the *rate of change of velocity*:

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt} \right) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right)$$

It is surprising that there are hardly any problems of mechanics that involve derivatives of the spatial coordinates w.r.t. time higher than the second .

We do not even have a special name for the 3rd derivative of the spatial coordinates w.r.t. time. Although intuitively we know that accelerations also change with time, but the rate of change of acceleration plays no role in mechanics.

A central concept of mechanics is the **mass** of objects.

We learn at school that mass has two manifestations: inertial mass and gravitational mass, and that the two are equal.
We accept this and forget what a difficult path lies behind these words.

What we need is a ***definition of mass*** that is ***independent of the laws of motion.***

Newton was obviously aware of the problem, for he begins his Principia with the following definition of mass:

Definition I
The quantity of matter is the measure of the same, arising from its density and bulk conjointly.

and he continues: It is this quantity … that is called **mass.**

The use of the word ***bulk*** in this way is old-fashioned: today we use the word ***volume*** instead.

Let us analyse Newton’s definition of mass.

At first sight it looks strange: we are used to define ***density*** in terms of ***mass*** and ***volume***, and Newton does it the other way round, he defines mass in terms of volume and density!

But then we see that his definition is ***operational***.

Indeed, consider a body that can be easily compressed. Snow is a good example, or a sponge or a pillow case full of feathers.

If you compress this body, then you say that its density is increased, but the amount of matter, *i.e.* its mass, remains unchanged.

Newton’s definition also implies that we can change the mass of the body without changing its density: we can do this by dividing the body into two parts. The masses of the parts will be in proportion to their volumes.

But Newton's definition does not help us when we want to compare the masses of bodies of different materials, such as the masses of balls of steel and of wood.

To make that comparison we need to understand a manifestation of mass, for instance the response of bodies to applied forces.

Responses of bodies to forces are of two kinds: a force can cause a ***deformation*** of a body or it can affect its ***state of motion***.

The latter case is the content of Newton's second law of mechanics. But that law also involves the mass. Therefore we must not use it to define force, lest we end up with a circular argument.

Before we can write down the second law in a logically meaningful way, we must give an **independent definition of force**, that is **not** itself **based on these laws**.

Force can be defined in a way that is independent of the laws of motion by considering systems in which forces are in ***equilibrium***.

The study of such systems is the content of ***statics,*** which is a branch of mechanics.

The laws of statics were well known even to the ancients. Suffice it to mention the ***laws of the lever***. In our time they are amply taught at school.

What we need to realise for the purpose of defining the concept of force is that objects, on which forces act, do get deformed.

Deformations can be made measurable and thereby serve as a quantitative measure of force.

An example of a body whose deformation can be made measurable is a spring.

We apply to the spring what we ***intuitively*** call force, the force of our muscles, and we observe that the spring is *extended* or *compressed*, depending on the direction of the force.

Then we say that the cause of the extension (or compression) of a spring is a force.

Next we suspend the spring from a rigid support and attach to its free lower end a body of regular shape, for instance a cylindrical piece of steel. We observe that the spring is extended and conclude that the steel cylinder exerts a force on the spring. We call that force the ***weight*** of the body.

We can ***calibrate*** the spring by attaching cylinders of the same material but different volume, and therefore of different mass if we accept Newton's definition of mass.

A series of such experiments will lead us to the conclusion that the weight W of a body is proportional to its mass m:

$$W \propto m$$

Then we suspend from the calibrated spring a body of a different material and thus measure its weight, and hence by comparing the weights of bodies of different materials we compare their masses.

Finally we need to define a ***standard of mass*** against which the masses of all bodies are compared.

The standard mass must be ***reproducible*** and ***durable***.

By international agreement, the standard of mass, called 1 kilogram, is the mass of a cylinder of a platinum-iridium alloy, kept in the French Bureau of Standards in Sevres, Paris.

Copies of this standard are kept in the bureaus of standards of all countries worldwide.

But note: what we have defined here is the ***gravitational mass***.

Newton's great achievement was to realise that the ***inertial mass*** of a body is identical with its gravitational mass.

Newton’s three laws of motion

Law I:
Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it.

Law II:
The change of motion is proportional to the motive force impressed and is made in the direction of the right line in which the force is impressed.

Law III:
To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

I have given here the three laws of mechanics in the English translation made from the Latin original by Andrew Motte in 1729, two years after Newton's death.

What is called ***motion*** in these laws is what today we call ***momentum***; it is the product of mass times velocity:

$$\vec{p} = m\vec{v} \tag{1}$$

The first law seems to follow from the second law as a particular case: the motion of a body on which no impressed force acts. So it looks rather redundant.

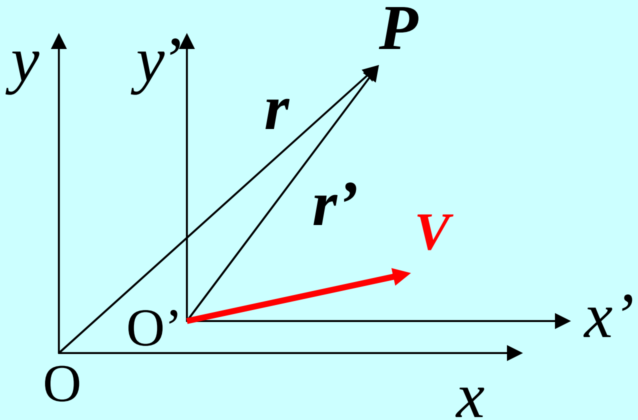
But in fact, the first law has a deep meaning: it is the statement that there exist reference frames in which a body persists in its state of rest or of uniform motion in the absence of external forces.

This must be compared with the state of motion of the same body, observed from a rotating reference frame: here the path of the body is curved.

A reference frame that fulfils the condition of Newton's first law is called ***inertial frame,*** and the property by which the body persists in its state of motion is called ***inertia.***

It follows then that any reference frame, that is in uniform motion with respect to a given inertial frame, is also an inertial frame.

Such frames are related by ***Galilean transformations***:



$$\vec{r} \rightarrow \vec{r}\,' = \vec{r} - \vec{V}t \qquad (2)$$

where $\vec{V}$ is the relative velocity between the two inertial frames, and is assumed to be constant.

It is a simple mathematical exercise to show that a point ***P*** that travels with a constant velocity in one frame travels also with a constant velocity in the other frame.

Let us now turn to Newton's second law: in mathematical form it is written as

$$\frac{d\vec{p}}{dt} = \vec{F} \qquad (3)$$

or in modern words (and remembering that this is true only in an inertial frame):

the rate of change of momentum of a body is equal to the force acting on it.

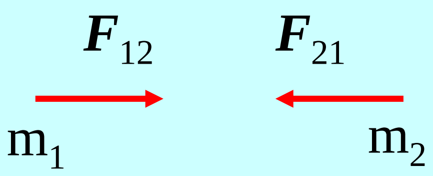
If the mass of the body is constant, then we can rewrite Newton's second law in the following form:

$$m\frac{d\vec{v}}{dt} = \vec{F} \quad \text{or indeed} \quad m\vec{a} = \vec{F} \qquad (3a)$$

The mass m in Eq. (3a) is the ***inertial mass*** of the body. The question of whether the inertial mass is identical with the gravitational mass has occupied many physicists for centuries. It was given its final form by ***Einstein*** in his ***equivalence principle***.

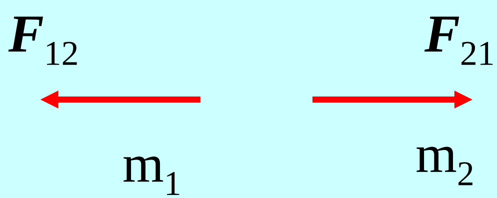
Lastly, the third law, the famous ***"action equals reaction".***

If we have two point-like bodies which interact with each other, then we can write the third law in the following mathematical form:

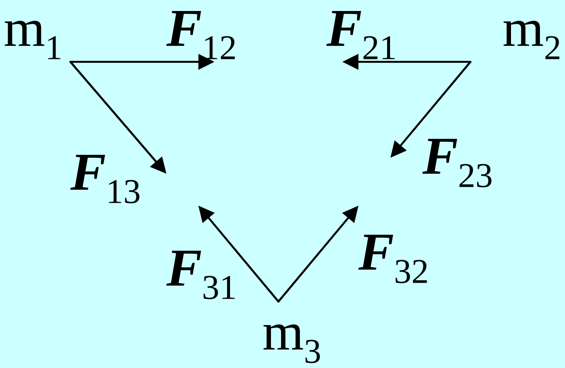


$$\vec{F}_{12} = -\vec{F}_{21} \tag{4}$$

The forces can be ***attractive***, as shown in the above figure, or they can be ***repulsive***, as shown in the next figure.



If there are three interacting particles in the system, then Newton's third law generalises naturally:



$$\vec{F}_{21} = -\vec{F}_{12}$$
$$\vec{F}_{23} = -\vec{F}_{32} \qquad (5)$$
$$\vec{F}_{31} = -\vec{F}_{13}$$

Generally, in a system of n interacting particles we have

$$\vec{F}_{ji} = -\vec{F}_{ij}, \quad i, j = 1, 2, ..., n; \quad j \neq i \qquad (6)$$

It follows from Eq. (6) that the sum of all forces of interaction between the particles of an n-particle system is zero:

$$\sum_{i \neq j = 1}^{n} \vec{F}_{ij} = 0 \qquad (7)$$

To describe the dynamics of a system of n interacting particles we must write down the equation of motion (3) for each particle:

$$\frac{d\vec{p}_1}{dt} = \vec{F}_{12} + \vec{F}_{13} + \ldots + \vec{F}_{1n}$$

$$\frac{d\vec{p}_2}{dt} = \vec{F}_{21} + \vec{F}_{23} + \ldots + \vec{F}_{2n} \qquad (8)$$

$$\ldots$$

$$\frac{d\vec{p}_n}{dt} = \vec{F}_{n1} + \vec{F}_{n2} + \ldots + \vec{F}_{n,n-1}$$

If we add these n equations, then we get

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \ldots + \frac{d\vec{p}_n}{dt} = \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n\right) = \frac{d\vec{P}}{dt} = 0 \qquad (9)$$

where $\boldsymbol{P}$ is the total momentum of the system.

We conclude that the total momentum $\boldsymbol{P}$ of a system of n interacting particles is constant:

$$\vec{P} = \text{constant} \qquad (10)$$

The equation we have found, Eq. (10), is a ***conservation law***.

A system of particles in which the only forces are forces of interaction between the particles, is called a ***closed system***.

We can therefore express the conservation law (10) by saying that

the total momentum of a closed system of particles is constant.

Let us apply a **Galilean transformation** (**GT**) to the reference frame in which our system has total momentum ***P.***

By Eq. (2) from slide 18 we have for the position of the *k*th particle

$$\vec{r}_k \rightarrow \vec{r}_k^{\,'} = \vec{r}_k - \vec{V}t \qquad (2)$$

and hence its velocity and momentum

$$\vec{v}_k \rightarrow \vec{v}_k^{\,'} = \vec{v}_k - \vec{V}; \qquad \vec{p}_k \rightarrow \vec{p}_k^{\,'} = \vec{p}_k - m_k\vec{V} \qquad (11)$$

and for the total momentum of the n particle system we get

$$\vec{P} \rightarrow \vec{P}^{'} = \vec{P} - M\vec{V} \qquad (12)$$

where M is the total mass of the system:

$$M = \sum_{k=1}^{n} m_k$$

Of particular interest is the case $\boldsymbol{P'} = 0$. Then we get from Eq. (12):

$$\vec{V} = \frac{\vec{P}}{M} = \frac{d}{dt}\left(\frac{\sum_k m_k \vec{r}_k}{M}\right) \qquad (13)$$

The vector

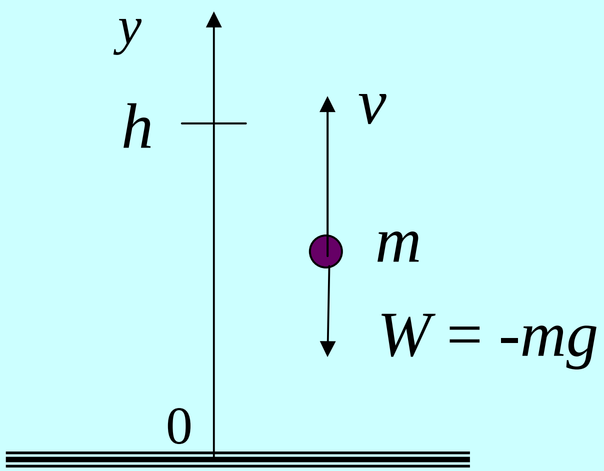
$$\vec{R} = \sum_k m_k \vec{r}_k \Big/ M \qquad (14)$$

is the radius vector of the ***centre of mass*** of the *n* particle system.

Conclusion: the GT (2) with velocity given by Eq. (13) takes us to a reference frame in which the centre of mass is at rest.

Such a frame is called ***centre of mass frame***.

1.2 Examples: projectile in constant gravitational field and in $1/r^2$ field



Point mass m is projected upwards with initial velocity v_0.

The force impressed on m is its weight W

Then the equation of motion is

$$m\frac{dv(t)}{dt} = -mg \qquad (15)$$

The masses on the left and right-hand sides are the inertial and gravitational masses, respectively, but we are following Galileo, Newton and Einstein in setting them equal. Therefore the mass cancels and we get

$$\frac{dv(t)}{dt} = -g \qquad (15a)$$

We have introduced g as a proportionality constant between mass and weight. Now we see that it has the dimension of an acceleration. It is called the ***acceleration due to gravity***.

g does not depend on any property of the body: it is valid ***universally***. The universality of g was experimentally established by Galileo.

Integrating Eq. (15a) over time from t_0 to t we get

$$v(t) = v_0 - g\left(t - t_0\right)$$

and we note that $v(t)=dy/dt$, and hence by integrating once more we get

$$y(t) = y_0 + v_0\left(t - t_0\right) - \frac{1}{2}g\left(t - t_0\right)^2$$

If the initial time and position are $t_0=0$, $y_0=0$, then we have

$$v\left(t\right) = v_0 - gt \quad \text{and} \quad y\left(t\right) = v_0 t - \frac{1}{2}gt^2 \qquad \text{(16a,b)}$$

The maximum height ***h*** to which the projectile can rise is the height at which the speed is zero; from Eq. (16a) we see that this is reached at the time

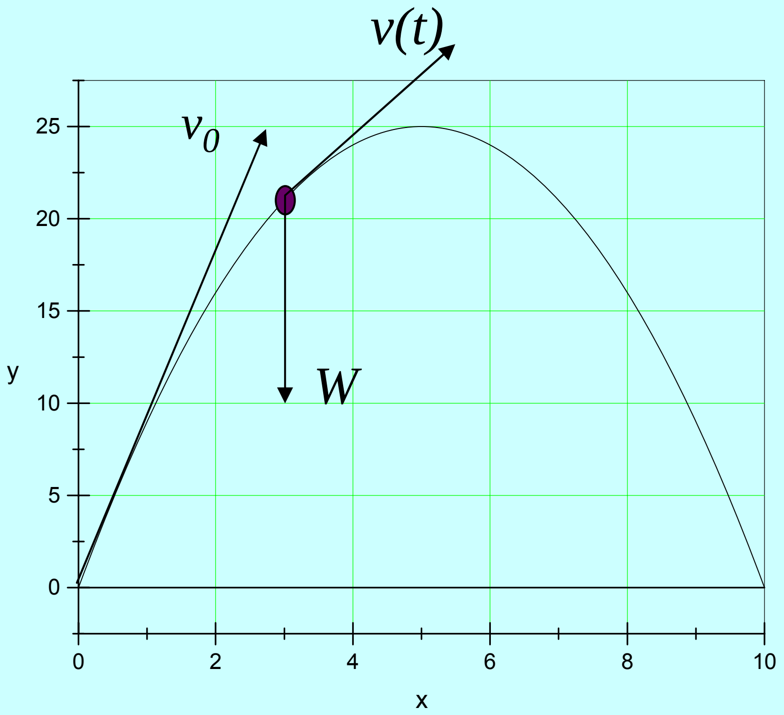
$$t_h = v_0 / g$$

and hence from Eq. (16b):

$$h = v_0 t_h - \frac{1}{2} g t_h^2 = \frac{1}{2} \frac{v_0^2}{g}$$

We can use this result to measure g. It is not going to be the most accurate experiment but it will give a rough answer and it should stimulate us to think of doing better, more accurate measurements of g.

Consider a projectile whose initial velocity is inclined to the horizontal.



If we choose the initial velocity to lie in the (x,y) plane, then the entire trajectory stays in the (x,y) plane, and we need consider only the equations of motion of the x and y components:

$$m\frac{dv_x(t)}{dt} = 0$$

$$m\frac{dv_y(t)}{dt} = -mg$$

The mass cancels, we do the integrations and then we get

$$v_x(t) = v_{x0}, \quad v_y(t) = v_{y0} - g(t - t_0)$$

and after a further integration we get

$$x(t) = x_0 + v_{x0}(t - t_0)$$

$$y(t) = y_0 + v_{y0}(t - t_0) - \frac{1}{2} g \left(t - t_0\right)^2$$

Let $t_0=0$, $x_0=0$ and $y_0=0$, hence

$$x(t) = v_{x0} t, \quad y(t) = v_{y0} t - \frac{1}{2} g t^2$$

We can find the range, *i.e.* the maximum distance the projectile travels over horizontal ground: after a straightforward calculation we get

$$x_{\max} = 2 \frac{v_{x0} v_{y0}}{g}$$

or, if we denote by α the angle which the initial velocity makes with The x axis, then

$$v_{x0} = v_0 \cos\alpha; \quad v_{y0} = v_0 \sin\alpha$$

and hence

$$\boxed{x_{\max} = \frac{v_0^2}{g}\sin 2\alpha}$$

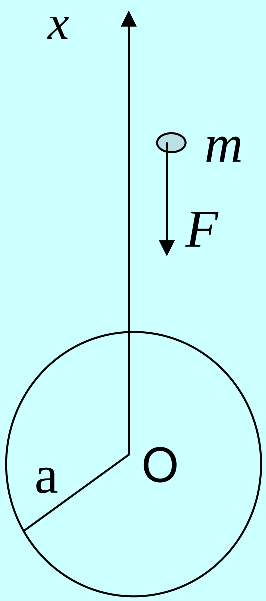
Thus we get the greatest range if the projectile is launched at an angle of 45 degrees to the horizontal.

Taking the approximate value of g = 10 m s^{-2}, we find that the projectile must be fired at roughly the speed of sound to reach 10 kilometres and roughly twice the speed of sound to reach 40 km.

But these results are no more than a very rough estimate because at such speeds the air resistance plays a significant role.

Next consider a projectile in the earth's gravitational field at a great height.

For simplicity, consider the case of a projectile of mass m launched radially.



Put the origin of the x axis at the centre of the earth, then the gravitational attraction at distances $x \geq a$, where a is the earth's radius, is given by

$$F = -G\frac{mM}{x^2}$$

where G is Newton's gravitational constant and M is the mass of the earth.

The equation of motion is

$$m\frac{d^2x}{dt^2} = -G\frac{mM}{x^2} \qquad (17)$$

The mass m of the projectile cancels, and we have

$$\ddot{x} = -G\frac{M}{x^2} \tag{18}$$

where from now on I shall write

$$\dot{x} = \frac{dx}{dt}, \qquad \ddot{x} = \frac{d^2x}{dt^2}, etc.$$

$\dot{y}, \ddot{y}, etc.$

For $x = a$, *i.e.* at the earth's surface, the right-hand side of Eq. (18) is the acceleration due to gravity g. Therefore we can rewrite (18) in the form of

$$\ddot{x} = -g\frac{a^2}{x^2} \tag{18a}$$

To solve this differential equation, we multiply by $\dot{x}$ and note that

$$\dot{x}\ddot{x} = \frac{d}{dt}\left(\frac{\dot{x}^2}{2}\right) \quad \text{and} \quad \frac{\dot{x}}{x^2} = -\frac{d}{dt}\left(\frac{1}{x}\right)$$

hence

$$\frac{d}{dt}\left(\frac{\dot{x}^2}{2}-g\frac{a^2}{x}\right)=0$$

and therefore

$$\frac{\dot{x}^2}{2}-g\frac{a^2}{x}=C=const. \qquad (19)$$

The constant C is a constant of integration. It is defined by the ***initial conditions*** which we are at liberty to choose.

Assume that initially, *i.e*. at $t = 0$, the projectile is at $x = \mathrm{a}$ and has initial velocity v_0, hence

$$C=\frac{1}{2}v_0^2-ga \qquad (19a)$$

Putting together Eqs. (19) and (19a) we get after obvious manipulations

$$v = \dot{x} = \sqrt{v_0^2 - 2ga + \frac{2ga^2}{x}} \qquad (20)$$

The velocity v must be real, therefore the radicand must be non-negative, thus

$$v_0^2 - 2ga + \frac{2ga^2}{x} \geq 0$$

or

$$v_0^2 \geq 2ga\left(1 - \frac{a}{x}\right) = 2ga\left(1 - \frac{a}{a+h}\right) = 2ga\frac{h/a}{1+h/a}$$

where h is defined by $x = a+h$. *i.e.* h is the height above the earth's surface.

To travel to a distance $h \to \infty$, *i.e.* to leave the earth's gravitational field, the projectile must have an initial velocity

$$\boxed{v_0 \geq \sqrt{2ga}}$$

The smallest initial velocity needed to leave the gravitational field of the earth is called ***escape velocity***. Putting in the numbers for the acceleration due to gravity and for the earth's radius we get

$$v_{esc} = 11.2\ \mathrm{km\,s^{-1}}$$

Recall the case of a projectile in a constant gravitational field. There we have found the height h to which the projectile can rise, given the initial velocity v_0:

$$h = \frac{1}{2}\frac{v_0^2}{g}$$

and this shows that the projectile cannot escape with any finite initial velocity.

Exercises and Problems for Selected Topics in Physics

Problem Sheet 1: classical mechanics

Standard notation is used in the following exercises.

1. Find the maximum height of a projectile above horizontal ground in constant gravitational field and ignoring air resistance in terms of its initial velocity and angle of launch.

Answer: $$h_{\max} = \frac{1}{2}\frac{v_0^2}{g}\sin^2\alpha$$

2. Find the maximum height of a projectile above horizontal ground in constant gravitational field and ignoring air resistance in terms of its range *R* and of the angle of launch.

Answer: $$h_{\max} = \frac{1}{4} R \tan\alpha$$