Mathematical Techniques
Part 5: Ordinary Differential Equations.

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Contents:

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1. Introduction.

An equation which specifies a relationship between a function, its argument and its derivatives of the first, second, etc. order is called a *differential equation*. Thus a differential equation could be of the form

$$y'(x) \equiv \frac{dy(x)}{dx} = f(x,y). \tag{1}$$

Here the highest order of derivative of the function $y(x)$ w.r.t. x is the first order. Therefore the differential equation is called differential equation *of the first order*.

In Newtonian mechanics one considers the position x of a particle of mass m as a function of time t: $x = x(t)$. If the force $F(x,t)$ acting on the particle is known, then one can write down a *second order* differential equation to find $x(t)$:

$$m\,\frac{d^2x(t)}{dt^2} = F(x,t) \tag{2}$$

which is Newton's second law of mechanics.

The main problem of the theory of differential equations is to find the unknown function which, substituted into the differential equation, turns it into an identity. Such a function is called the *solution* or *integral* of the differential equation.

It is possible that a differential equation has no solution. More usual is that the differential equation has infinitely many solutions. Even the simplest kind of first order differential equation has usually an infinite manifold of solutions. Thus, consider the following first order differential equation:

$$y'(x) = f(x) \tag{3}$$

This is a particular case of Deq. (1) which is particularly simple because the function $f(x)$ depends only on x and does not depend on y. In this case we can immediately find a solution as long as $f(x)$ is integrable:

$$y(x) = \int f(x)\,dx + C \tag{4}$$

where C is an integration constant which is not defined by the differential equation. This is the source of the nonuniqueness of the solution of the differential equation: we can give the integration constant any value, and each time we specify a different value of C we are also selecting a different solution of the differential equation.

Later on we shall see that the solution of a second order differential equation has two integration constants, and more generally, an nth order differential equation has n integration constants.

There are several generalizations of the simple differential equations written down above. One such generalization arises if we have several functions of the same argument, $x(t)$, $y(t)$ and $z(t)$, say, and correspondingly three simultaneous differential equations, which could be of the first, second etc. order. Thus, for instance, we can have the following system of simultaneous differential equations:

$$\frac{d^2x(t)}{dt^2} = F(x, y, z, t), \qquad \frac{d^2y(t)}{dt^2} = G(x, y, z, t), \qquad \frac{d^2z(t)}{dt^2} = H(x, y, z, t). \tag{5}$$

Such simultaneous equations arise for instance in mechanics where they describe the motion of a particle in three-dimensional space under the influence of a time-dependent force field, whose x, y and z components are the right-hand sides of Eqs. (5).

Another generalization arises when the unknown function depends on more than one argument. For instance the propagation of electromagnetic waves in free space is described by an equation of the following form:

$$\frac{\partial^2 A(x, y, z, t)}{\partial t^2} = c^2 \left[\frac{\partial^2 A(x, y, z, t)}{\partial x^2} + \frac{\partial^2 A(x, y, z, t)}{\partial y^2} + \frac{\partial^2 A(x, y, z, t)}{\partial z^2} \right]$$

Because of the appearance of partial derivatives this is called a *partial* differential equation. In this course we will consider only ordinary differential equations. Partial differential equations are treated from a mathematical point of view in more advanced mathematics courses; they are also discussed in courses on electromagnetism and, of course, in quantum mechanics.

2. Ordinary differential equations of the first order.

2.1) *Basic definitions.*
The general form of a first order differential equation is

$$y'(x) = f(x, y) \tag{6}$$

where $y'(x)$ denotes the first derivative of y w.r.t. x. Here it is assumed that the function $f(x, y)$ is known. We can therefore calculate $f(x, y)$, and hence y', at any point in the cartesian (x, y) plane. Since we know that the first derivative of $y(x)$ is, geometrically, its slope, we can make a useful plot where we show at a large number of points the corresponding slopes. Such a plot is called a *direction field.*

Consider the following example. Let

$$y' = \frac{x - y}{x + y} \tag{7}$$

then we get the direction field as shown in Fig. 1.

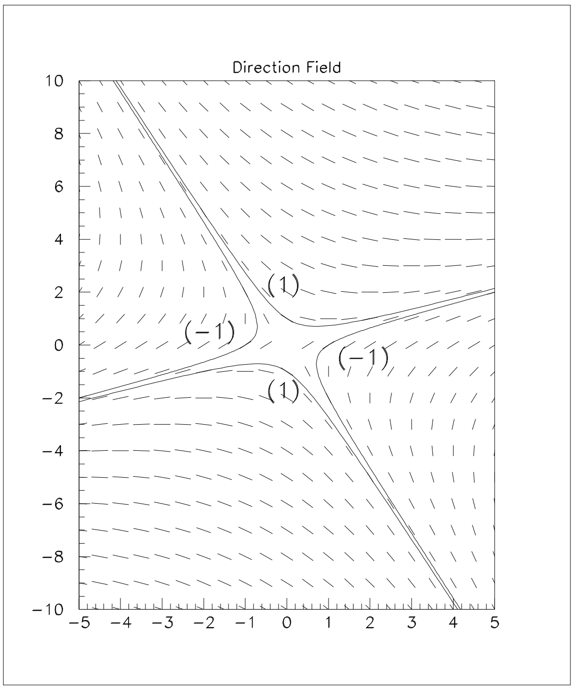

Figure 1: Direction field of differential equation $y' = (x - y)/(x + y)$

Our aim is to find a solution (or all solutions) of the differential equation. As was stated in the Introduction, a solution of the differential equation (7) is a function $y(x)$ which, substituted into the differential equation, turns it into an identity. We can convince ourselves by substitution that the solution of Deq. (7) is given by

$$y = -x \pm \sqrt{2x^2 + C} \tag{8}$$

where C is an arbitrary integration constant. To draw a curve of the function we must specify C. Such a curve is called an *integral curve* of the differential equation. Take, for instance, $C = 1$. The corresponding integral curve is labeled (1) on Fig. 7.1. The two branches of the curve correspond to the two signs in Eq. (8). We can see that all along the integral curve the short arrows of the direction field coincide with the tangents of the curve. With $C = -1$ we get the integral curve labeled (-1), and again we see that the arrows give the directions of the tangents along the curve. Conversely we can say that, given the direction field, we can construct an approximate integral curve graphically starting from a specified point by following the directions of the arrows.

In the above examples we have chosen the integration constant and x in order to get an integral curve. We can proceed differently and choose a pair of values for x and y and hence calculate the corresponding value of the integration constant. Such a pair of values of x and y are called *initial conditions*. The initial conditions completely define an integral curve. Choosing different initial conditions we get, in general, different integral curves.

The solution of a first order differential equation, which contains the arbitrary integration constant, is called the *general solution*. When the integration constant is given a specific value, for instance by specifying the initial conditions, the solution is called a *particular solution*.

Exercise 2. Verify by substitution that $y = Ce^x - x - 1$ is the general solution of the differential equation

$$y' = y + x.$$

Given the initial conditions (i) $y = 1$ at $x = 0$, (ii) $y = 0$ at $x = 0$, (iii) $y = -1$ at $x = 0$, calculate C and draw the corresponding integral curves for $x \in [-2, 2]$.

2.2) *Some types of exactly soluable first order differential equations.*

2.2.1) Separable equations. The simplest type of differential equations are the separable equations. They are of the general form

$$y' = f(x)g(y) \tag{9}$$

Dividing by $g(y)$ and multiplying by dx we get

$$\frac{dy}{g(y)} = f(x)\,dx \tag{10}$$

which can be solved by quadrature:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C \tag{11}$$

Many techniques to solve first order differential equations have as their guiding principle some strategy of reducing the equation to a separable form.

Exercise 3. Solve the following differential equations:
(i) $xy' = y \ln x$;
(ii) $\sin(y)\,y' = \sin(x + y) - \sin y \cos x$;
Answer: (i) $y = C \exp[(\ln x)^2]$; (ii) $y = \arccos Ce^{\cos x}$.

2.2.2) Linear differential equations.

A differential equation is called *linear* if it contains the unknown function $y(x)$ and its derivative $y'(x)$ linearly, i.e. if it is of the form

$$p(x)y' + q(x)y = f(x) \tag{12}$$

where $p(x)$, $q(x)$ and $f(x)$ are known functions of x.

In the particular case, when $f(x) = 0$, the linear differential equation is called *homogeneous*; if $f(x) \neq 0$ it is called *inhomogeneous*.

Consider first the homogeneous linear differential equation

$$p(x)y' + q(x)y = 0 \tag{13}$$

Multiplying by dx, dividing by $p(x)$ and rearranging we get

$$\frac{dy}{y} = -\frac{q(x)}{p(x)}\,dx \tag{14}$$

and hence, integrating, we get

$$\ln y = -\int \frac{q(x)}{p(x)}\,dx + C' \tag{15}$$

where we have denoted the integration constant by C'. If we now exponentiate and set $C' = \ln C$, we get the general solution in the following form:

$$y = C \exp\left\{-\int \frac{q(x)}{p(x)}\,dx\right\} \tag{16}$$

This result shows that one can always find the general solution of a homogeneous linear differential equation by quadrature: all that remains to do, given the functions $p(x)$ and $q(x)$, is to take the integral of their ratio.

Exercise 4. Find the general solutions of the following linear homogeneous differential equations:
(i) $y' + xy = 0$,
(ii) $xy' + y = 0$,
(iii) $\sin(x)\,y' + \cos(x)\,y = 0$,
Answer: (i) $y = C\exp(-x^2/2)$; (ii) $y = C/x$; (iii) $y = C\exp(-\ln\sin x)$.

Let us next consider the class of inhomogeneous linear differential equations

$$p(x)y' + q(x)y = f(x) \tag{17}$$

A function $y(x)$, which is a solution of this equation but does not contain an integration constant, is called a *particular* solution of the differential equation. We shall denote the particular solution by $y_p(x)$.

We can show that the general solution of the inhomogeneous linear differential equation (17) is the sum of a particular solution y_p and the general solution of the homogeneous differential equation, obtained by setting $f(x) = 0$. This latter function is called *complementary* function, denoted $y_c(x)$. Thus the statement is that the general solution of Eq. (17) is of the form

$$y = y_c + y_p$$

We verify our statement by substituting y into the differential equation. On the left-hand side we get

$$p(x)y' + q(x)y = p(x)(y_c' + y_p') + q(x)(y_c + y_p) = [p(x)y_c' + q(x)y_c] + [p(x)y_p' + q(x)y_p]$$

and we see that the expression in the first bracket vanishes by definition of the complementary function and the expression in the second bracket is identically equal to $f(x)$. Since, furthermore, y has one integration constant it follows that it is the general solution.

We have previously dealt with the problem of solving homogeneous linear differential equations. Therefore we know how to find the complementary function. It remains to find a particular solution y_p. Sometimes this can be done by inspection, but there is also a systematic way of finding a particular solution known as the method of *variation of the constant*. This method consists of writing down the complementary function in the form of Eq. (16) and then replacing the integration constant C by a new unknown function $u(x)$, i.e. we put

$$y_p = u(x) \exp\left\{-\int \frac{q(x)}{p(x)}\, dx\right\}$$

To find $u(x)$ we substitute y_p into Eq. (17). For the derivative of y_p we get

$$y_p' = \left[u' - \frac{q(x)}{p(x)} u(x)\right] \exp\left\{-\int \frac{q(x)}{p(x)}\, dx\right\}$$

and hence, after substitution and simplification, we are left with

$$u'(x) = \frac{f(x)}{p(x)} \exp\left\{-\int \frac{q(x)}{p(x)}\, dx\right\}$$

The remarkable result is that the function $u(x)$ has cancelled and the equation contains only its derivative and known functions of x. We can therefore get $u(x)$ by one further quadrature. It is important at this stage to be aware that we need only a *particular* solution. That means that we must *not* introduce another integration constant.

Example. Find the general solution of the inhomogeneous linear differential equation

$$y' + xy = x.$$

Solution: in the previous exercise we have found the complementary function to be $y_c(x) = C\mathrm{e}^{-\frac{1}{2}x^2}$. To find a particular solution we vary the integration constant by replacing C with $u(x)$, i.e. we set

$$y_p(x) = u(x)\, \mathrm{e}^{-\frac{1}{2}x^2}$$

hence

$$y_p'(x) = [u'(x) - xu(x)]\, \mathrm{e}^{-\frac{1}{2}x^2}$$

and therefore

$$y_p'(x) + xy_p(x) = [u'(x) - xu(x)]\, \mathrm{e}^{-\frac{1}{2}x^2} + xu(x)\, \mathrm{e}^{-\frac{1}{2}x^2}$$

and we see that the terms linear in $u(x)$ cancel and we get the following differential equation for $u(x)$:

$$u'(x) = x\mathrm{e}^{\frac{1}{2}x^2}$$

hence

$$u(x) = \int x\mathrm{e}^{\frac{1}{2}x^2}\, dx = \mathrm{e}^{\frac{1}{2}x^2}$$

and therefore finally

$$y_p(x) = 1$$

and we note, of course, that this result could have been guessed by inspection of the differential equation. We write down our final result, i.e. the general solution, as the sum of the complementary function and the particular solution:

$$y = C\mathrm{e}^{-\frac{1}{2}x^2} + 1.$$

Example. Find the general solution of the inhomogeneous linear differential equation[1]

$$y' + ay = b \sin \omega x.$$

Solution: The complementary function is easily found to be $y_c(x) = C \exp(-ax)$. We find the particular solution of the inhomogeneous equation by variation of the constant, i.e. we set

$$y_p(x) = u(x)e^{-ax}$$

Upon substitution into the given differential equation and rearrangement we get

$$u'(x) = be^{ax} \sin \omega x$$

and hence

$$u(x) = b \int e^{ax} \sin \omega x \, dx$$

The integral can be taken by integration by parts (twice!), hence

$$y_p(x) = \frac{ab}{a^2 + \omega^2} \left(\cos \omega x + \frac{\omega}{a} \sin \omega x \right)$$

and hence the general solution

$$y(x) = Ce^{-ax} + \frac{ab}{a^2 + \omega^2} \left(\cos \omega x + \frac{\omega}{a} \sin \omega x \right)$$

The integration constant C can be found from the initial conditions. If, for instance, $y(x) = 0$ at $x = 0$, then $C = b\omega / \left(a^2 + \omega^2 \right)$ and the integral is

$$y(x) = \frac{b}{a^2 + \omega^2} \left(\omega e^{-ax} + a \sin \omega x - \omega \cos \omega x \right)$$

or

$$y(x) = \frac{b\omega}{a^2 + \omega^2} e^{-ax} + \frac{b}{\sqrt{a^2 + \omega^2}} \sin(\omega x - \gamma)$$

where $\tan \gamma = \omega / a$.

For small x, such that $\max(ax, \omega x) \ll 1$, we get $y = \dfrac{1}{2} b\omega x^2 + \mathcal{O}\left(\max((ax)^3, (\omega x)^3) \right)$. For values of x such that $ax \gg 1$ the decaying exponential is negligible and we have a purely sinusoidal oscillation, shifted in phase by γ relative to the applied force.

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[1] this linear differential equation describes the forced oscillations in an a.c. circuit. We can identify x with the time t, $y(x)$ with the current $I(t)$, and set $a = R/L$, $b = E/L$, where R is the ohmic resistance, L is the selfinduction and E is the peak value of the applied e.m.f. The right hand side is the forcing term, assumed to be periodic with frequency $f = \omega / 2\pi$.

Exercise 5. Find the general solutions of the following differential equations:
(i) $\cos(x)\, y' - \sin(x)\, y = \cos^2(x)$;
(ii) $y' - 2xy = x - x^3$;
(iii) $y' + \dfrac{x}{1+x^2} y = \dfrac{1}{x(1+x^2)}$;
Answer: (i) $y = \dfrac{C+x}{2\cos x} + \dfrac{1}{2}\sin x$; (ii) $y = C\mathrm{e}^{x^2}$; (iii) $y = \dfrac{C + \ln|x/(1+\sqrt{1+x^2})|}{\sqrt{1+x^2}}$.

2.2.3) Homogenous differential equations.
At the beginning of this section we have written the first order differential equation in the general form of Eq. (6). If we multiply this equation by dx it takes the form of $dy - f(x,y)dx = 0$, and if we multiply this by $Q(x,y)$ and define $P(x,y) = Q(x,y)f(x,y)$, then we get the symmetric form

$$P(x,y)dx + Q(x,y)dy = 0 \tag{18}$$

which is equivalent with Eq. (6).

The differential equation (18) is called *homogeneous* if $P(x,y)$ and $Q(x,y)$ are *homogeneous functions* of x and y of equal order.

A function $f(x,y)$ of two variables x and y is homogeneous of order m if

$$f(\lambda x, \lambda y) = \lambda^m f(x,y) \tag{19}$$

Examples:

$$f(x,y) = ax + by \qquad \text{is homogeneous of order 1,} \tag{20}$$
$$f(x,y) = x^3 + y^3 + 2xy^2 - 3x^2y \qquad \text{is homogenous of order 3,} \tag{21}$$
$$f(x,y) = \frac{ax+by}{\alpha x + \beta y} \quad \text{with} \quad a,\, b,\, \alpha,\, \beta \neq 0 \qquad \text{is homogeneous of order 0.} \tag{22}$$

Homogeneous differential equations can always be reduced to a seperable form by setting $y = xu(x)$.

Indeed, if we substitute $y = xu(x)$ into Eq. (18), then, assuming $P(x,y)$ and $Q(x,y)$ to be homogeneous of order m, we get

$$P(x,xu)dx + Q(x,xu)(udx + xdu) = x^m \left[P(1,u)dx + uQ(1,u)dx + xQ(1,u)du\right] = 0$$

or with $p(u) = P(1,u) + uQ(1,u)$ and $q(u) = Q(1,u)$

$$p(u)dx + xq(u)du = 0$$

and hence

$$\frac{dx}{x} + \frac{q(u)}{p(u)}du = 0$$

Occasionally it is convenient to discuss homogeneous differential equations in the from of Eq. (6), *i.e.*

$$y' = f(x,y) \tag{23}$$

Comparing with Eq. (18) we see that the differential eqaution (23) is homogeneous if the function $f(x, y)$ is homogeneous of order 0. It follows that

$$f(x, y) = f\left(1, \frac{y}{x}\right)$$

and hence, setting again $y = xu(x)$,

$$xu' + u = \varphi(u) \tag{24}$$

where $\varphi(u) = f(1, u)$. Thus in this approach we have again reduced the homogeneous differential equation to a seperable differential equation.

Example:

$$(x + 3y)dx - (3x + y)dy = 0$$

Here the coefficients of dx and dy are both homogeneous functions of order 1, *i.e.* the differential eqaution is homogeneous. Setting $y = xu$ we get

$$(1 + 3u)dx - (3 + u)(udx + xdu) = (1 - u^2)dx - x(3 + u)du = 0$$

hence

$$\frac{dx}{x} = \frac{3 + u}{1 - u^2}\, du$$

and carrying out the quadrature, going back from u to y and simplifying we get the result

$$(x - y)^2 = C(x + y), \qquad C = \text{ constant}$$

2.2.4) Exact differential equations.

For a discussion of exact differential equations it will be convenient again to use the symmetric form of Eq. (18), *i.e.*

$$P(x, y)dx + Q(x, y)dy = 0 \tag{25}$$

Let us assume that there is a function $U(x, y)$ such that its differential coincides with the left-hand side of Eq. (25), i.e.

$$dU(x, y) = P(x, y)dx + Q(x, y)dy = 0. \tag{26}$$

If this is the case, then the differential equation is called *exact*.

Now, by the chain rule we have

$$dU(x, y) = \frac{\partial U(x, y)}{\partial x}\, dx + \frac{\partial U(x, y)}{\partial y}\, dy \tag{27}$$

and since the differentials dx and dy are independent, Eqs. (26) and (27) together imply that

$$\frac{\partial U(x, y)}{\partial x} = P(x, y) \qquad \text{and} \qquad \frac{\partial U(x, y)}{\partial y} = Q(x, y) \tag{28}$$

and since the mixed second partial derivatives of $U(x, y)$ are independent of the order in which the derivatives are taken, i.e.

$$\frac{\partial^2 U(x, y)}{\partial x \partial y} = \frac{\partial^2 U(x, y)}{\partial y \partial x}$$

it follows that

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x} \tag{29}$$

To summarise, we have found that, if the differential equation (25) is an exact equation, then the coefficients $P(x,y)$ and $Q(x,y)$ satisfy Eq. (29). The converse is also true: if the coefficients $P(x,y)$ and $Q(x,y)$ satisfy Eq. (29), then Eq. (25) is an exact differential equation. Therefore Eq. (29) is a simple test for a differential equation to be exact.

Once we have established that a differential equation is an exact one, it is easy to find its general solution by two quadratures. Indeed, taking the first one of Eqs. (28) we get upon integration

$$U(x,y) = \int P(x,y)\,dx + \phi(y) \tag{30}$$

where instead of an integration constant we had to add an unknown function of y whose partial derivative w.r.t. x is, of course, equal to nought. If we now take the partial derivative w.r.t. y and use the second of Eqs. (28), then we get

$$\frac{\partial U(x,y)}{\partial y} = Q(x,y) = \frac{\partial}{\partial y}\int P(x,y)\,dx + \frac{d\phi(y)}{dy}$$

from which we can find $\phi(y)$. We substitute this into Eq. (30) and note that, because of $dU(x,y) = 0$, we have $U(x,y) = C$. Thus the general solution of the exact differential equation is

$$U(x,y) = \int P(x,y)\,dx + \phi(y) = C \tag{31}$$

Example. Given the differential equation

$$\frac{2x}{y^3}dx + \frac{y^2 - 3x^2}{y^4}dy = 0$$

let us check whether it is exact. To do that we identify the coefficients of dx and dy with the functions $P(x,y)$ and $Q(x,y)$ of the general case, respectively, i.e. we set

$$P(x,y) = \frac{2x}{y^3}, \qquad Q(x,y) = \frac{y^2 - 3x^2}{y^4}$$

and we apply the test (29):

$$\frac{\partial P(x,y)}{\partial y} = -\frac{6x}{y^4}, \qquad \frac{\partial Q(x,y)}{\partial x} = -\frac{6x}{y^4}$$

i.e. the differential equation is exact. We therefore proceed to find the function $U(x,y)$:

$$U(x,y) = \int P(x,y)\,dx + \phi(y) = \int \frac{2x}{y^3}\,dx + \phi(y) = \frac{x^2}{y^3} + \phi(y)$$

Next we differentiate w.r.t. y and equate to $Q(x,y)$:

$$\frac{\partial U(x,y)}{\partial y} = -3\frac{x^2}{y^4} + \frac{d\phi(y)}{dy} = \frac{y^2 - 3x^2}{y^4}$$

hence

$$\frac{d\phi(y)}{dy} = \frac{1}{y^2}$$

i.e.

$$\phi(y) = -\frac{1}{y}$$

where we did not need an integration constant since that appears when we set $U(x, y)$ equal to a constant:

$$U(x, y) = \frac{x^2}{y^3} - \frac{1}{y} = C \qquad (32)$$

Multiplying by y^3 and rearranging we get the solution finally in the form of

$$Cy^3 + y^2 = x^2$$

Differentiating Eq. (32) we can verify that this is indeed the general solution of the given differential equation.

Exercise 6. Show that the following differential equations are exact and hence find their general solutions:
(i) $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$;
(ii) $\dfrac{xdx + ydy}{\sqrt{1 + x^2 + y^2}} + \dfrac{ydx - xdy}{x^2 + y^2} = 0$;
(iii) $\left(\dfrac{1}{x} - \dfrac{y^2}{(x-y)^2}\right) dx + \left(\dfrac{x^2}{(x-y)^2} - \dfrac{1}{y}\right) dy = 0$;
Answer: (i) $x^3 + 3x^2y^2 + y^4 = C$; (ii) $\sqrt{1 + x^2 + y^2} + \arctan\dfrac{x}{y} = C$;
(iii) $y = \dfrac{xy}{x-y} + \ln\dfrac{x}{y} = C$.

2.2.4) Integrating factor.

Exact differential equations are exceptional. In most cases the coefficient functions $P(x, y)$ and $Q(x, y)$ will not satisfy the condition (29). In this case the differential equation (25) is called *nonexact.*

To solve a nonexact differential equation one proceeds by finding an *integrating factor* $\mu(x, y)$, i.e. such a function which, when multiplied into the differential equation, turns it into an exact one.

Thus assume that the differential equation

$$P(x, y)\, dx + Q(x, y)\, dy = 0 \qquad (33)$$

is nonexact. Let us multiply the equation by a function $\mu(x, y)$ and demand that the resulting equation be exact:

$$\mu(x, y)P(x, y)\, dx + \mu(x, y)Q(x, y)\, dy = 0 \qquad (34)$$

This requirement implies that the coefficients of dx and dy must satisfy a condition of the form of Eq. (29) but with $P(x, y)$ and $Q(x, y)$ replaced by $\mu(x, y)P(x, y)$ and $\mu(x, y)Q(x, y)$, respectively, i.e.

$$\frac{\partial\mu(x, y)P(x, y)}{\partial y} = \frac{\partial\mu(x, y)Q(x, y)}{\partial x} \qquad (35)$$

or

$$\frac{\partial \ln \mu(x,y)}{\partial y} P(x,y) - \frac{\partial \ln \mu(x,y)}{\partial x} Q(x,y) = \frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \tag{36}$$

At first sight it seems that we have gained nothing because we have replaced the problem of solving an ordinary differential equation by the more difficult problem of solving a partial differential equation for the unknown function $\mu(x,y)$. However there are several particular cases where the problem simplifies and immediately leads to the solution of the differential equation (33).

(i) Assume that we can find an integrating factor μ which is a function of x only. Then $\partial \mu(x)/\partial y = 0$ and Eq. (36) simplifies to

$$\frac{d \ln \mu(x)}{dx} = \frac{1}{Q(x,y)} \left[\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x} \right] \tag{37}$$

The expression on the r.h.s. is a known function; if the assumption that μ was a function of x only was correct, then the r.h.s. must also be a function of x only. In this case Eq. (37) gives us the integrating factor $\mu(x)$ with a single quadrature.

Example. Consider the differential equation

$$(y \tan x + \cos x)\, dx - dy = 0$$

i.e. $P(x,y) = y \tan x + \cos x$ and $Q(x,y) = -1$. Taking the difference of the partial derivatives as required by Eq. (29) we get

$$\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x} = \tan x$$

i.e. the differential equation is nonexact. We need an integrating factor. Assume that the integrating factor is a function of x only, i.e. that $\mu = \mu(x)$. By Eq. (37) this is the case if

$$\frac{d \ln \mu(x)}{dx} = \frac{1}{Q(x,y)} \left[\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x} \right]$$

has a right-hand side which is a function of x only. Evaluating the r.h.s. we get

$$\frac{d \ln \mu(x)}{dx} = -\tan x$$

which implies that our assumption was correct. We therefore proceed:

$$\ln \mu(x) = -\int \tan x\, dx = \ln \cos x$$

or

$$\mu(x) = \cos x$$

At this stage we do not need an integration constant which will arise when we solve the given differential equation.

Having found an integrating factor we proceed to solve the differential equation. We check that $\mu P dx + \mu Q dy = 0$ is an exact equation and carry on as in section 2.2.3. It is left to the reader to find the solution which is (cf. Exercise 5 (i))

$$y = \frac{1}{2} \left[\frac{x + C}{\cos x} + \sin x \right].$$

(ii) Assume that we can find an integrating factor μ which is a function of y only. Then $\partial\mu(y)/\partial x = 0$ and Eq. (36) simplifies to

$$\frac{d\ln\mu(y)}{dy} = \frac{1}{P(x,y)}\left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right] \tag{38}$$

The expression on the r.h.s. is a known function; if the assumption that μ was a function of y only was correct, then the r.h.s. must also be a function of y only. In this case Eq. (38) gives us the integrating factor $\mu(y)$ with a single quadrature.

Example. Consider the differential equation

$$y^2\,dx + (x - 2xy - y^2)\,dy = 0$$

i.e. $P(x,y) = y^2$ and $Q(x,y) = x - 2xy - y^2$. Taking the difference of the partial derivatives as required by Eq. (29) we get

$$\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x} = 4y - 1$$

i.e. the differential equation is nonexact. We need an integrating factor. Assume that the integrating factor is a function of y only, i.e. that $\mu = \mu(y)$. By Eq. (38) this is the case: we get

$$\frac{d\ln\mu(y)}{dy} = \frac{1}{P(x,y)}\left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right] = \frac{1-4y}{y^2}$$

and we get the integrating factor

$$\mu(y) = y^{-4}\mathrm{e}^{-1/y}$$

We multiply the given differential equation by the integrating factor and check that we have got an exact differential equation:

$$\frac{\mathrm{e}^{-1/y}}{y^2}\,dx + \frac{\mathrm{e}^{-1/y}}{y^4}\left(x - xy - y^2\right)\,dy = 0$$

and we verify:

$$\frac{\partial}{\partial y}\left(\frac{\mathrm{e}^{-1/y}}{y^2}\right) - \frac{\partial}{\partial x}\left[\frac{\mathrm{e}^{-1/y}}{y^4}\left(x - xy - y^2\right)\right] = \left(\frac{1}{y^4} - \frac{2}{y^3}\right)\mathrm{e}^{-1/y} - \frac{\mathrm{e}^{-1/y}}{y^4}(1-2y) = 0$$

and hence proceed to solve this differential equation. The solution is

$$U(x,y) = \left(\frac{x}{y^2} - 1\right)\mathrm{e}^{-1/y} = C$$

To verify our result we differentiate to get

$$dU(x,y) = \frac{dx}{y^2}\mathrm{e}^{-1/y} + \left(-\frac{2x}{y^3} + \frac{x}{y^4} - \frac{1}{y^2}\right)\mathrm{e}^{-1/y}\,dy = 0$$

and dropping the exponential factor and multiplying by y^4 we get the given differential equation.

(iii) Similar to the above two are the cases of integrating factors which are either a function of $(x+y)$ or of $(x-y)$. If these methods fail one has to resort to more complicated approaches which are outside the scope of this course.

Problems:

1. Show that $\mu = (xP + yQ)^{-1}$ is an integrating factor if $(xP + yQ) \neq 0$ and P and Q are homogeneous functions of the same order.

2. Show that $\mu = (xP - yQ)^{-1}$ is an integrating factor if $(xP - yQ) \neq 0$ and $P = yg(xy)$, $Q = xh(xy)$.

3. Show that $\mu = m(x)n(y)$ is an integrating factor if $\partial P/\partial y - \partial Q/\partial x$ can be brought in the form of $PY(y) - QX(x)$.

4. Show that $\mu = (P^2 + Q^2)^{-1}$ is an integrating factor if $\partial P/\partial x = \partial Q/\partial y$ and $\partial P/\partial y = -\partial Q/\partial x$.

In these lecture notes I have discussed only a few methods of solving first order differential equations. To get a complete picture of all known methods a textbook on differential equations should be consulted. However, whereas it is useful to study such methods, the practicing scientist will resort either to a monograph, that contains systematically collected differential equations together with their solutions, for instance, "Differential Equations" by E. Kamke, which contains more than 500 first order differential equations and many more second order equations. In this computerised age there are also computer packages available which are capable of solving most known differential equations analytically. Such packages are, for instance, Maple, Mathematica and REDUCE. Computer packages have the advantage that they have graphics facilities which allow for almost instant display of the integral curves.

3. Ordinary differential equations of the second order.

3.1 Preliminary Remarks.

A surprisingly large number of differential equations encountered in physical applications are of second order. This makes the study of second order equations particularly important.

For instance, in classical mechanics the one-dimensional motion of a particle is described by the second order differential equation

$$m\ddot{x}(t) = F(\dot{x},\, x,\, t) \tag{39}$$

where $x(t)$ is the position of the particle of mass m at time t, $\dot{x} = dx/dt$ its velocity, $\ddot{x} = d^2x(t)/dt^2$ its acceleration and $F(\dot{x},\, x,\, t)$ is the force on the particle.

A particular case is the differential equation of forced, damped harmonic oscillations:

$$\ddot{x}(t) + 2\beta\dot{x} + \omega^2 x = f\cos(\Omega t) \tag{40}$$

where ω is the *natural* frequency of the oscillator, 2β is the friction coefficient and Ω is the driving frequency. One can check by substitution that the solution is given by

$$x(t) = Ae^{-\beta t}\cos(\omega t + \alpha) + B\cos(\Omega t + \phi) \tag{41}$$

where A and α are integration constants to be determined by initial conditions, and B and ϕ are given by

$$B = \frac{f}{\sqrt{(\omega^2 - \Omega^2)^2 + 4\beta^2\Omega^2}}, \qquad \phi = \arctan\frac{2\beta\Omega}{\Omega^2 - \omega^2} \tag{42}$$

Later on in these notes we will come back to this important differential equation and study it in detail. Here we want to concentrate on one aspect of the form of the solution, namely the appearance of *two* integration constants. This should be seen alongside the situation we had in the case of first order differential equations, whose solutions have *one* integration constant. To discuss the question of integration constants in general terms, let us write down the general form of the solutions of differential equations:

$$\begin{aligned} y &= F(x;C) \qquad \text{in the case of 1st order Deqs.} \\ y &= F(x;C_1,C_2) \qquad \text{in the case of 2nd order Deqs.} \end{aligned}$$

which are also called *one-parametric* and *two-parametric* family of functions (or curves), respectively.

If we write down the one-parametric function and its first derivative, we get the following two equations:

$$y = F(x;C) \qquad \text{and} \qquad y' = \frac{dF(x;C)}{dx}$$

From these two equations we can eliminate the constant C and get

$$y' = \Phi(y,x)$$

i.e. a first-order differential equation. If we take the two-parametric function, then we need two more relations to eliminate the integration constants. Thus we get

$$y = F(x;C_1,C_2), \qquad y' = \frac{dF(x;C_1,C_2)}{dx}, \qquad y'' = \frac{d^2F(x;C_1,C_2)}{dx^2}$$

and from these three equations we can eliminate the two constants and get

$$y'' = \Phi(y',y,x)$$

i.e. a second-order differential equation. This procedure can be extended to an arbitrary number of constants C_1, C_2, ..., C_n: by writing down this function and its derivatives up to the nth order we get $n+1$ relations from which the constants can be eliminated to leave us with one equation, relating the n derivatives (and x) to each other, i.e. we get an nth order differential equation

$$y^{(n)} = \Phi(y^{(n-1)},\ldots,y',y,x)$$

Conversely we conclude that the general solution of an nth order differential equation has n integration constants. If these constants are given particular values, then the solution is called a *particular* solution.

3.2 Initial conditions.

Usually it is more convenient to fix not the integration constants but the values of the function and its derivatives at a certain value of the argument x. This is because these constants have a more intuitive physical meaning than the integration constants, which come out of the mathematical wash. In this case one says that one specifies the *initial conditions*. We have previously shown in the case of first order differential equations that the two methods are equivalent. Let us now repeat the argument for the case of second order differential equations.

Thus let us assume that we have found the general solution of the second order differential equation

$$y'' = F(y',y,x) \tag{43}$$

which is of the form

$$y = y(x; C_1, C_2) \tag{44}$$

and let us assume that we are given the following initial conditions:

$$\text{for} \qquad x = x_0: \qquad y = y_0 \quad \text{and} \quad y' = y'_0 \tag{45}$$

where x_0, y_0 and y'_0 are fixed numerical values. Then we can write down the following two equations:

$$y(x_0, C_1, C_2) = y_0, \qquad \text{and} \qquad y'(x_0, C_1, C_2) = y'_0 \tag{46}$$

from which we can express the integration constants in terms of x_0, y_0 and y'_0.

Example: Consider the differential equation of a particle of mass m freely falling in the gravitational force of the Earth:

$$m\ddot{x}(t) = -gm$$

whose general solution is

$$x(t) = C_2 + C_1 t - \frac{1}{2}gt^2$$

If we assume that the particle was released at time $t = 0$ from rest at a height h, we can write down the initial conditions in the following form:

$$\text{IC:} \qquad \text{at} \quad t = 0 \qquad x = h \quad \text{and} \quad \dot{x} = 0$$

which give us the following two equations to find the integration constants:

$$C_2 = h, \qquad C_1 = 0$$

3.3 Second order linear differential equations.

Second order linear differential equations play an important role in physics. We have already seen that they describe oscillations, which alone form a rich and important field of applications, especially if it is recognised that oscillations of electric circuits are described by the same type of equation. Other examples are the differential equation of a membrane which, when reduced to one dimension, becomes a second order linear differential equation. Similarly the time independent Schrödinger equation of quantum mechanics leads in many cases to this type of differential equation.

The general form of second order linear differential equations is

$$p(x)y'' + q(x)y' + r(x)y = f(x) \tag{47}$$

where $p(x)$, $q(x)$, $r(x)$ and $f(x)$ are known functions of x

If $f(x) = 0$ identically, the equation is called *homogeneous*, otherwise it is *inhomogeneous*.

The simplest case of linear differential equations is the case of *constant coefficients*, i.e. when the functions $p(x)$, $q(x)$ and $r(x)$ are constants.

3.3.1 Linear second order differential equations with constant coefficients.

The differential equation which we are now going to study is of the form

$$y'' + ay' + by = f(x) \tag{48}$$

where a and b are constants. If also $f(x) = 0$ we have the homogeneous linear second order differential equation with constant coefficients

$$y'' + ay' + by = 0 \tag{49}$$

Its solution can be found by making the substitution

$$y = e^{\lambda x} \tag{50}$$

where λ is a constant to be found by demanding the function to satisfy the differential equation. If we differentiate the exponential function we get

$$y' = \lambda e^{\lambda x}, \qquad y'' = \lambda^2 e^{\lambda x}$$

and upon substitution of these into the differential equation we get

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

and since the exponential function is everywhere nonzero, we can drop the factor $\exp \lambda x$ and get a quadratic equation in the unknown constant λ:

$$\lambda^2 + a\lambda + b = 0 \tag{51}$$

which is the *auxiliary* equation; this equation has two roots:

$$\lambda_{1,2} = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right) \tag{52}$$

giving us two solutions of the differential equation:

$$y_1 = e^{\lambda_1 x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x} \tag{53}$$

These two functions are *particular* solutions. So the question is now how to find the *general* solution which, as we know, must contain two integration constants. Here the following observation helps: if $y(x)$ is a solution of Deq. (48), then $Cy(x)$ is also a solution, if C is a constant. Indeed, if we substitute $Cy(x)$ into the differential equation, we get

$$C(y'' + ay' + by) = 0 \tag{54}$$

and since by assumption $y(x)$ is a solution, therefore the expression in brackets is identically equal to nought. Thus, to go back to our situation, we can construct two solutions of the form $C_1y_1(x)$ and $C_2y_2(x)$.

The second observation which will help us is the following: if $y_1(x)$ and $y_2(x)$ are two solutions of a linear homogeneous differential equation, then their sum $y = y_1(x) + y_2(x)$ is also a solution. Indeed, if we substitute y into the differential equation (48), then we get on the l.h. side

$$y'' + ay' + by = \frac{d^2}{dx^2}(y_1 + y_2) + a\frac{d}{dx}(y_1 + y_2) + b(y_1 + y_2) \tag{55}$$

and because of the linearity of the differential operation this gives

$$[y_1'' + ay_1' + by_1] + [y_2'' + ay_2' + by_2] \tag{56}$$

and both of the expressions in square brackets are separately equal to nought because y_1 and y_2 are by assumption solutions of the differential equation.

Combining our two observations we can say that, if $y_1 = \exp(\lambda_1 x)$ and $y_2 = \exp(\lambda_2 x)$ are two solutions of the given differential equation, then

$$y(x) = C_1 \exp(\lambda_1 x) + C_2 \exp(\lambda_2 x) \tag{57}$$

is also a solution. Moreover, since $y(x)$ contains two arbitrary integration constants, it is the general solution of the Deq. (48).

3.3.2 Discussion of the solutions.

In order to have a physical picture in mind we will associate the case of oscillations with the differential equation under discussion. It is then natural to use the usual notation, considering the independent variable to be the time t; the function $x(t)$ is then the displacement, and instead of a and b we shall denote the constants by 2β and ω_0. Thus our differential equation is

$$\ddot{x}(t) + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{58}$$

Let us repeat the procedure of finding the general solution. We begin by the substitution

$$x(t) = e^{\lambda t} \tag{59}$$

which yields the equation

$$\left(\lambda^2 + 2\beta\lambda + \omega_0^2\right) e^{\lambda t} = 0 \tag{60}$$

and since $\exp(\lambda t) \neq 0$ we have the auxiliary equation

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0 \tag{61}$$

whose two roots are

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{62}$$

We must now distinguish three caes:

(i) light damping: $\beta^2 - \omega_0^2 < 0$. In this case the roots are complex. We make this explicit by defining the positive constant $\omega^2 = \omega_0^2 - \beta^2$, hence

$$\lambda_{1,2} = -\beta \pm i\omega, \qquad i = \sqrt{-1} \tag{63}$$

The general solution is now

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = e^{-\beta t}\left(C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right) \tag{64}$$

or, using the Euler formula $\exp(\pm iz) = \cos z \pm i \sin z$, we get

$$x(t) = e^{-\beta t}\left[(C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t\right] = e^{-\beta t}\left(a\cos\omega t + b\sin\omega t\right) \tag{65}$$

where in the last step we have defined two new constants: $a = C_1 + C_2$ and $b = i(C_1 - C_2)$. An alternative form of the solution is obtained by setting $a = A\cos\alpha$ and $b = -A\sin\alpha$, hence

$$x(t) = Ae^{-\beta t}\cos(\omega t + \alpha) \tag{66}$$

In the absence of resistance $\beta = 0$, and the solution takes on the well known form of simple harmonic oscillation

$$x(t) = A\cos(\omega t + \alpha) \tag{67}$$

where A is the amplitude and α is the initial phase.

(ii) heavy damping: $\beta^2 - \omega_0^2 > 0$. In this case both roots of the auxiliary equation are real. If the coefficient of friction β is positive, as is usually the case, both roots are less than nought, i.e. the general solution is a superposition of two exponential functions, which both decrease with time, albeit at a different rate. If we define the real constant $\Lambda = \sqrt{\beta^2 - \omega_0^2}$, we can write the general solution in the form of

$$x(t) = e^{-\beta t}\left(C_1 e^{\Lambda t} + C_2 e^{-\Lambda t}\right) \tag{68}$$

or in terms of the hyperbolic functions $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ and $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$:

$$x(t) = e^{-\beta t}\left(A\cosh \Lambda t + B\sinh \Lambda t\right) \tag{69}$$

(iii) critical damping: $\beta^2 - \omega_0^2 = 0$. This case is of considerable practical importance since a critically damped system settles down after a disturbance most quickly. One therefore attempts to employ critical damping in many measuring devices, such as scales or ballistic galvanometers.

Mathematically we have the problem, that the two solutions, $x_1 = \exp \lambda_1 t$ and $x_2 = \exp \lambda_2 t$, coincide and therefore their linear superposition becomes $x = (C_1 + C_2)\exp \lambda t$, where I have set $\lambda = \lambda_1 = \lambda_2$. Thus our solution has only one integration constant $C = C_1 + C_2$, i.e. it is *not* the general solution. The question is how to get a second solution which would be different from the one we have already got, so that we can write down the general solution.

The answer is that such a solution is $x_2 = t\exp \lambda t$. This can be checked by substitution[2]: we have

$$\dot{x}_2 = (1 + \lambda t)e^{\lambda t}, \qquad \text{and} \qquad \ddot{x}_2 = (2\lambda + \lambda^2 t)e^{\lambda t}$$

hence, with $\lambda = -\beta$ and $\omega_0^2 = \beta^2$,

$$\ddot{x}_2 + 2\beta\dot{x}_2 + \omega_0^2 x_2 = [(-2\beta + \lambda^2 t) + 2\beta(1 - \beta t) + \beta^2 t]e^{-\beta t} = 0$$

Having got a second solution we can write down the general solution:

$$x(t) = C_1 x_1 + C_2 x_2 = (C_1 + C_2 t)e^{-\beta t}$$

3.3.3 Linear independence.

In the previous section we have come across situations where we had either two distinct solutions, which we could use to construct the general solution, or two coinciding solutions, in which case we needed to find a second, distinct solution. More generally, we can have the case that we get solutions which look different, but which none the less cannot be used to make the general solution.

Consider the following example. Assume that we need to solve a third order differential equation. In this case we are looking for three solutions. Now assume that we have found solutions $\sin^2 x$, $\cos^2 x$ and 1. If we superimpose them linearly we get $C_1 \sin^2 x + C_2 \cos^2 x + C_3 = (C_1 + C_3)\sin^2 x + (C_2 + C_3)\cos^2 x$, in other words, what we hoped to be the general solution has only two integration constants, $(C_1 + C_3)$ and $(C_2 + C_3)$, and so we must work harder to get the general solution.

[2] two different derivations of this result are given in Appendix 1

What this example is teaching us is that it is not enough for the solutions to *look* different. What we need are solutions which are *linearly independent*. To explain what linear independence means, let us first consider the opposite case of linear dependence: given a set of n functions $y_1(x)$, $y_2(x)$, …, $y_n(x)$ one says that they are linearly dependent if any one of them can be expressed by a linear superposition of all the other functions, for instance if

$$y_1(x) = c_2 y_2(x) + \ldots + c_n y_n(x)$$

with similar relations for all other functions. Since none of the functions is in any way distinguished, it is preferable to rewrite the above relation in a symmetric form:

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0$$

Thus one says that, if such a relation exists where not all superposition coefficients are zero, then the functions are linearly dependent. Conversely, linear independence is defined by the following statement:

Definition. The functions $y_1(x)$, $y_2(x)$, …, $y_n(x)$ are linearly independent if

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0$$

if and only if $c_1 = c_2 = \ldots = c_n = 0$.

Consider two differentiable functions $y_1(x)$ and $y_2(x)$ which may or may not be linearly independent. Let us write down the following equation which should hold identically in x:

$$c_1 y_1(x) + c_2 y_2(x) = 0 \tag{70}$$

Differentiating Eq. (70) w.r.t. x we get

$$c_1 y_1'(x) + c_2 y_2'(x) = 0 \tag{71}$$

Equations (70) and (71) can be rewritten in matrix form:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \tag{72}$$

The determinant

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

is called the *Wronskian* of the functions y_1 and y_2. We can convince ourselves that the coefficients c_1 and c_2 are equal to zero if the Wronskian is not equal to nought. Indeed, if we multiply Eq. (70) by y_2' and Eq. (71) by y_2 and subtract, then we get

$$c_1 \left(y_1 y_2' - y_2 y_1' \right) = c_1 W = 0$$

and hence $c_1 = 0$ if $W \neq 0$. Similarly $c_2 = 0$. Since $c_1 = c_2 = 0$ is the condition of linear independence of y_1 and y_2 we conclude that these functions are linearly independent if $W \neq 0$.

Let us apply this to the solutions of the second-order linear differential equation with constant coefficients:

$$y_1 = e^{\lambda_1 x}, \qquad y_2 = e^{\lambda_2 x}$$

The Wronskian of this pair of solutions is

$$W = \det \begin{pmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{pmatrix} = (\lambda_2 - \lambda_1)\, e^{(\lambda_1+\lambda_2)x}$$

and we see that $W \neq 0$ if $\lambda_2 - \lambda_1 \neq 0$. On the other hand, if the two roots coincide, then $W = 0$ as it must be if the solutions are not linearly independent.

The concept of the Wronskian can be generalised to the case of n linearly independent functions. We again write

$$c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) = 0$$

and differentiate $(n-1)$ times. This gives the following system of simultaneous equations:

$$\begin{aligned} c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) &= 0 \\ c_1 y_1'(x) + c_2 y_2'(x) + \ldots + c_n y_n'(x) &= 0 \\ \cdots \quad \cdots \quad & \cdots \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \ldots + c_n y_n^{(n-1)}(x) &= 0 \end{aligned}$$

or in matrix form:

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix} = 0 \qquad (73)$$

and we see that the condition of linear independence of the functions $y_i(x)$, $i = 1, 2, \ldots, n$ is equivalent with the statement

$$W = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \neq 0$$

3.3.4 Inhomogenous linear differential equation.

The inhomogeneous linear differential equation is of the general form

$$p(x)y'' + q(x)y' + r(x)y = f(x) \qquad (74)$$

(cf. Eq. (47)). This equation can be solved in two steps: (i) find the general solution of the corresponding homogeneous equation, i.e. of the equation with the same coefficient functions $p(x)$, $q(x)$ and $r(x)$ but with $f(x)$ replaced by zero; this solution is called the *complementary function* denoted by $y_c(x)$. (ii) Find a particular solution $y_p(x)$ of the inhomogeneous equation, i.e.

$$p(x)y_p'' + q(x)y_p' + r(x)y_p = f(x) \qquad \text{identically} \qquad (75)$$

Then the general solution of the inhomogeneous linear differential equation is the sum of the complementary function and $y_p(x)$:

$$y(x) = y_c(x) + y_p(x)$$

Indeed, if we substitute $y(x)$ into the differential equation and reorder terms, we get on the l.h. side

$$[p(x)y_p'' + q(x)y_p' + r(x)y_p] + [p(x)y_c'' + q(x)y_c' + r(x)y_c] \tag{76}$$

and we see that the first pair of square brackets gives $f(x)$ and the second pair is identically zero. Finally, since $y_c(x)$ - and therefore also $y(x)$ - contains two arbitrary integration constants, it follows that $y(x)$ is the general solution of the inhomogeneous linear differential equation.

Our result remains true also for constant coefficients, i.e. for the differential equation

$$y'' + ay' + by = f(x) \tag{77}$$

(cf. Eq. (48)). Since for this particular case we have a general way of finding the complementary function, all that remains to do is to find the particular solution $y_p(x)$.

In some simple cases the particular solution can be guessed.

(i) Consider the case $f(x) = f_0 = constant$. Then, if we put $y_p = C$ with an unknown constant C, we have $y_p' = 0$ and $y_p'' = 0$, hence

$$bC = f_0 \tag{78}$$

from which we get $C = f_0/b$.

(ii) Next consider the linear function $f(x) = Cx$. If we put the particuler integral equal to a linear function $y_p = \alpha + \beta x$ with unknown coefficients α and β, then we get $y_p' = \beta$, $y_p'' = 0$, hence

$$a\beta + b(\alpha + \beta x) = Cx \tag{79}$$

and comparing coefficients we get the two equations

$$a\beta + b\alpha = 0, \qquad \text{and} \qquad b\beta = C \tag{80}$$

hence $\beta = C/b$, $\alpha = -a\beta/b = -aC/b^2$.

(iii) A similar procedure leads to the particular solution also in the case of $f(x) = Cx^2$: we put $y_p = \alpha + \beta x + \gamma x^2$, substitute, compare coefficients, and hence get

$$\gamma = C/b, \qquad \beta = -2aC/b^2 \qquad \text{and} \qquad \alpha = 2C(a^2 - b)/b^3 \tag{81}$$

The above cases can be quite easily extended to general quadratic polynomials, i.e. to r.h. sides of the form $f(x) = c_1 + c_2x + c_3x^2$.

(iv) Now consider $f(x) = f_0 \exp sx$ where f_0 and s are arbitrary constants. Then, if we put $y_p(x) = A \exp sx$ with unknown constant A, we get after substitution

$$A\left(s^2 + as + b\right)e^{sx} = f_0 e^{sx} \tag{82}$$

hence, after cancellation of the exponential function,

$$A = \frac{f_0}{s^2 + as + b} \tag{83}$$

In this derivation it was not important whether s was real or imaginary. The latter case is of particular interest in the mathematical treatment of oscillations. Thus, let us put $s = i\Omega$. Then we get

$$A = \frac{f_0}{b - \Omega^2 + ia\Omega} \tag{84}$$

and we can represent A in terms of its modulus $|A|$ and phase α:

$$|A| = \frac{f_0}{\sqrt{(b - \Omega^2)^2 + a^2\Omega^2}}, \qquad \alpha = \frac{\operatorname{Im} A}{\operatorname{Re} A} = \frac{a\Omega}{\Omega^2 - b} \tag{85}$$

(v) The case of a sinusoidal inhomogeneous term can be reduced to the case just considered. Thus let us return to the differential equation of the forced, damped harmonic oscillation which we have mentioned in the introduction, Eq. (40):

$$\ddot{x}(t) + 2\beta\dot{x} + \omega_0^2 x = f\cos(\Omega t) \tag{86}$$

There are two ways of dealing with this differential equation. The hard way is to seek the particular integral in the form of $x_p(t) = A\sin(\Omega t + \alpha)$. The simple way is to write down a second equation:

$$\ddot{y}(t) + 2\beta\dot{y} + \omega_0^2 y = f\sin(\Omega t) \tag{87}$$

and to consider x and y to be the real and imaginary parts of a new, complex function $z = x + iy$. Then, if we multiply Eq. (87) by the imaginary unit i and add it to Eq. (86), we get

$$\ddot{z}(t) + 2\beta\dot{z} + \omega_0^2 z = f\exp(i\Omega t) \tag{88}$$

whose particular solution we can take from the previous example with obvious changes of notation: $z_p(t) = A\exp(i\Omega t)$ with

$$A = \frac{f}{\omega_0^2 - \Omega^2 + 2i\beta\Omega} \tag{89}$$

or, if we put $A = |A|\exp(i\phi)$,

$$|A| = \frac{f}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\beta^2\Omega^2}}, \qquad \phi = \arctan\frac{2\beta\Omega}{\Omega^2 - \omega_0^2} \tag{90}$$

The particular solution of the given differential equation (86) is found by taking the real part of z:

$$x = \operatorname{Re} z = |A|\operatorname{Re} \mathrm{e}^{i(\Omega t+\phi)} = |A|\cos(\Omega t + \phi)$$

to which we must add the complementary function in order to get the general solution. However, as we have seen previously, the complementary function dies out exponentially due to the damping term $2\beta\dot{x}$, and at times $t \gg 1/\beta$ it is negligible. The particular solution, on the other hand, is a sinusoidal oscillation with an amplitude $|A|$ that is proportional to the strength f of the forcing term and depends on the parameters ω_0 and β of the oscillator and on the driving frequency Ω. The latter dependence is of particular interest: especially for small damping β the amplitude has a sharp maximum near $\Omega = \omega_0$. The occurrence of such a maximum is called a *resonance*, and the frequency Ω_r that corresponds to the maximum is called the *resonance frequency*. The phase shift ϕ between the driving force and the response also depends on Ω. For $\Omega = \omega_0$ we get $\phi = \pi/2$ whereas far from the resonance the phase shift is small.

Appendix 1: *On the derivation of a second, linearly independent solution of 2nd order linear homogeneous differential equations in the case of degeneracy.*

In our discussion of 2nd order linear homogeneous differential equations with constant coefficients we have considered three cases of the roots of the auxiliary equation. In one of these cases the two roots coincided. We have written down the solution and verified it. This procedure may seem unsatisfactory. Therefore we give here simple derivations of the formula.

First derivation: Let us restate the problem. The differential equation is

$$\ddot{x}(t) + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{91}$$

and the two solutions are

$$x_1(t) = e^{\lambda_1 t}, \qquad x_2(t) = e^{\lambda_2 t}$$

with

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

The problem arose for $\beta^2 - \omega_0^2 = 0$, when the two roots coincide. This is called *degeneracy* of the solutions.

Let us assume that $\beta^2 - \omega_0^2 \neq 0$. Then the roots are distinct and we have the two linearly independent solutions x_1 and x_2. We can construct two new linearly independent solutions from x_1 and x_2:

$$x_+ = \frac{1}{2}(x_1 + x_2), \qquad \text{and} \qquad x_- = \frac{1}{2}(x_1 - x_2)$$

In the limit of $\beta \to \omega_0$, which we are interested in, we have

$$\lim_{\beta\to\omega_0} x_+ = x_1, \qquad \lim_{\beta\to\omega_0} x_- = 0$$

so in the case under consideration only x_+ is useful. To construct the second solution we recall that we can multiply a solution by a constant. Let us define as before $\Lambda = \sqrt{\beta^2 - \omega_0^2}$; for $\beta \to \omega_0$ we have $\Lambda \to 0$. Now multiply x_- by $1/\Lambda$ and then take the limit $\Lambda \to 0$, then the second solution becomes

$$x_2 = \lim_{\Lambda\to 0} \frac{1}{\Lambda} x_- = \lim_{\Lambda\to 0} \frac{1}{2\Lambda} e^{-\beta t}\left(e^{\Lambda t} - e^{-\Lambda t}\right) = te^{-\beta t}$$

which is the desired result.

Second derivation: In case the first derivation appears artificial, consider an approach whose idea is similar to that of the variation of the constant employed in solving inhomogeneus first order linear differential equations. Thus consider the solution $x = \exp(\lambda t)$, which remains a solution after multiplication by a constant. Let us denote this by x_1, *i.e.*

$$x_1 = ce^{\lambda t}, \qquad c = \text{constant}$$

and assume the second solution to be of the form of $x_2 = u(t)\exp\lambda t$ with an unknown function $u(t)$. Differentiating x_2 we get

$$\begin{aligned} \dot{x}_2 &= \dot{u}\, e^{\lambda t} + \lambda u\, e^{\lambda t} \\ \ddot{x}_2 &= \ddot{u}\, e^{\lambda t} + 2\lambda\dot{u}\, e^{\lambda t} + \lambda^2\, u\, e^{\lambda t} \end{aligned}$$

and upon substitution into the differential equation, and recalling that $\lambda = -\beta = -\omega_0$, we get

$$\ddot{u} = 0$$

hence

$$u = c_1 + c_2 t$$

If we multiply this by x_1 and absorb the constant c in c_1 and c_2, then we see that we have found the general solution

$$x = (c_1 + c_2\, t)e^{-\beta t}$$

Appendix 2: **Note on numerical integration of differential equations.**

There are numerous methods of numerical integration of differential equations. The simplest method is due to Leonhard Euler. Write the differential equation in the form of

$$dy = f(x, y)\, dx,$$

replace the differentials dx and dy by the finite differences Δx and Δy and correspondingly replace the differential equation by the difference equation

$$\Delta y = f(x, y)\, \Delta x$$

which is an approximation to the given differential equation. The difference equation can be solved exactly by iteration, starting from some point (x_0, y_0). This solution is an approximate solution of the differential equation.

Let us see how Euler's method works in practice. It is usual to set Δx to a constant value, e.g. $\Delta x = h$. We start at the point (x_0, y_0) and find $\Delta y_0 = f(x_0, y_0)h$ and hence $x_1 = x_0 + h$ and

$$y_1 = y_0 + \Delta y_0.$$

Then we calculate $x_2 = x_1 + h$ and $\Delta y_1 = f(x_1, y_1)h$ and hence

$$y_2 = y_1 + \Delta y_1,$$

and we carry on with the iteration to get at the nth step $x_n = x_{n-1} + (n-1)h$, $\Delta y_{n-1} = f(x_{n-1}, y_{n-1})h$ and hence

$$y_n = y_{n-1} + \Delta y_{n-1}.$$

Iteration is terminated when x or y move outside the range of interest.

Example. Consider the differential equation

$$y' = \frac{1}{2}xy \tag{92}$$

which has the general solution $y = C \exp(x^2/4)$. With initial conditions $y = 1$ for $x = 0$ we have the particular integral $y = \exp(x^2/4)$. Now let us solve this differential equation numerically for the same initial conditions using Euler's method. Let us put $h = \Delta x = 0.1$. Then we can tabulate our work as shown in Table 1.

Table 1: Iterative solution of DEq. (92) by Euler's method

n	x_n	y_n	$0.5x_ny_n$	Δy_n	y_{n+1}
0	0	1	0	0	1
1	0.1	1	0.05	0.005	1.0050
2	0.2	1.0050	0.1005	0.0101	1.0151
3	0.3	1.0151	0.1523	0.0152	1.0303
4	0.4	1.0303	0.2061	0.0206	1.0509
5	0.5	1.0509	0.2627	0.0263	1.0772
6	0.6	1.0772	0.3232	0.0323	1.1095
7	0.7	1.1095	0.3883	0.0388	1.1483
8	0.8	1.1483	0.4593	0.0459	1.1942
9	0.9	1.1942	0.5374	0.0537	1.2479
10	1.0	1.2479	–	–	–

We have carried out all the numerical work to four decimal places to avoid an accumulation of rounding errors. It is left as an exercise for the reader to plot the approximate integral curve using the table and to show on the same graph the curve for the exact solution. The error made in the numerical solution is less than four percent at $x = 1$ and smaller below $x = 1$. A better accuracy, at the expense of more calculations, can be obtained by taking smaller steps h.

There are many improvements of Euler's method known. The basic idea of these is contained in the Runge-Kutta method. We shall not discuss these methods, but the interested reader is referred to the rich literature on numerical methods, notably the excellent book *Numerical Recipes* by W.H. Press, S.A. Teucholsky, W.T. Vettering and B.P. Flannery.

However, most of these methods have one deficiency: they fail when the integral curve has a vertical slope at some point and the iteration takes you to that point.[3] You don't need a perverse differential equation to run into that situation. Take, for instance, the differential equation

$$y' = (x - y)/(x + y)$$

which has this problem all along the line $y = -x$. In a case like this the following modification of Euler's method is always successful: instaed of demanding a constant step length in the x direction, we demand a constant step in the direction of the integral curve.

To implement this idea in an algorithm, let us rewrite the differential equation $y' = f(x, y)$ in symmetric form:

$$P(x, y)\, dx = Q(x, y)\, dy$$

which we replace again by its approximation in differenced form:

$$P(x, y)\, \Delta x = Q(x, y)\, \Delta y$$

The relationship between Δx and Δy is preserved if we put

$$\Delta x = h\, Q(x, y)/D(x, y), \qquad \Delta y = h\, P(x, y)/D(x, y), \quad \text{where} \quad D(x, y) = \sqrt{P^2(x, y) + Q^2(x, y)}$$

[3]another way of describing such an integral curve is to say that it represents a double-valued function: to every value of x there are two values of y. Obviously, there can be more complicated situations when the integral of the differential equation is a multivalued function. This would need an extension of the method discussed here.

and we note that the step along the integral curve is now given by

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2} = h = \text{constant}.$$

The algorithm for solving the pair of difference equations is the following: choose the initial point $(x_0,\, y_0)$, then calculate

$$\Delta x_0 = h\, Q(x_0, y_0)/D(x_0, y_0), \qquad \Delta y_0 = h\, P(x_0, y_0)/D(x_0, y_0)$$

and hence

$$x_1 = x_0 + \Delta x_0, \qquad y_1 = y_0 + \Delta y_0$$

then iterate, getting in the nth step

$$\Delta x_n = h\, Q(x_{n-1}, y_{n-1})/D(x_{n-1}, y_{n-1}), \qquad \Delta y_n = h\, P(x_{n-1}, y_{n-1})/D(x_{n-1}, y_{n-1})$$

and hence

$$x_n = x_{n-1} + \Delta x_{n-1}, \qquad y_n = y_{n-1} + \Delta y_{n-1}$$

We can see that this method avoids the problem of a vertical slope. Indeed, a vertical slope at some point (x, y) implies that $\Delta x = 0$ while $\Delta y \neq 0$. With the present method there is no problem because we make just another step of length h vertically, and at the new point along the integral curve we will usually have a slope other than a vertical one.

Example. Consider the differential equation $y' = (x - y)/(x + y)$ or in symmetric form

$$(x - y)\, dx - (x + y)\, dy = 0$$

Using the method of section 2.2.3 it is easily checked that this is an exact differential equation. The solution is

$$x^2 - y^2 - 2xy = C \qquad \text{or, solved for } y, \quad y = -x \pm \sqrt{2x^2 - C}$$

which is the equation of a family of hyperbolæ. The solution has real values of y if $x^2 \geq C/2$, which is satisfied for all nonpositive values of C. For positive values of C there are no real solutions if $x \in (-C/2, C/2)$. Let us choose the IC: $y_0 = -1$ at $x_0 = -5$, hence $C = 14$. We get the integral curve shown in Fig. 7.2, which is one branch of a hyperbola with vertical tangent at $x = -\sqrt{7} \approx -2.646$. Within the accuracy of graph plotting the exact integral curve coincides with that obtained by numerical integration.

The details of the calculation are shown in Table 2; a step length of $h = 0.01$ was chosen and the results were printed for every 50th iteration step. One can see that at first x is increasing, but after it reaches its maximum value of -2.64 it changes direction; this is also where $Q(x, y)$ is changing its sign. The function $y(x)$ is increasing monotonically. The numerical work was done in a FORTRAN program of only a few lines.

Table 2: Iterative solution of differential equation $(x-y)dx-(x+y)dy=0$

n	x	y	$P(x,y)$	$Q(x,y)$	$D(x,y)$
1	-5.00	-1.00	-4.00	-6.00	7.21
50	-4.60	-0.72	-3.88	-5.32	6.58
100	-4.20	-0.42	-3.79	-4.62	5.97
150	-3.83	-0.08	-3.74	-3.91	5.41
200	-3.48	0.28	-3.76	-3.20	4.94
250	-3.18	0.68	-3.86	-2.51	4.60
300	-2.94	1.11	-4.06	-1.83	4.45
350	-2.77	1.58	-4.36	-1.19	4.52
400	-2.68	2.07	-4.75	-0.60	4.79
450	-2.64	2.57	-5.22	-0.07	5.22
500	-2.66	3.07	-5.73	0.42	5.75
550	-2.71	3.57	-6.28	0.86	6.34
600	-2.79	4.06	-6.85	1.27	6.97
650	-2.89	4.55	-7.44	1.66	7.63
700	-3.01	5.04	-8.05	2.03	8.30
750	-3.13	5.52	-8.66	2.39	8.98
800	-3.27	6.00	-9.28	2.73	9.67
850	-3.42	6.48	-9.90	3.07	10.4
900	-3.57	6.96	-10.5	3.39	11.1
950	-3.72	7.43	-11.2	3.71	11.8
1000	-3.88	7.91	-11.8	4.03	12.5