

Representation of Functions by Taylor Series

Notes by W.B. von Schlippe, October 2003

It is frequently useful to represent functions by power series. Thus, for example, the trigonometric function $\sin(x)$ can be shown to be represented by the series

$$\sin(x) = 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} \pm \dots$$

In these notes it will be shown how such representations can be obtained in a general way. We shall derive the MacLaurin expansion formula and its generalisation, the Taylor expansion for arbitrary functions.

1.) Consider a function $y = f(x)$ which is continuous and differentiable in the closed interval $[a, b]$.¹ Then one can prove the following theorem, due to Lagrange:

Mean Value Theorem: If the function $y = f(x)$ is continuous in $[a, b]$ and has a continuous derivative in this interval, then there exists at least one point c between a and b such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad (1)$$

For a rigorous proof I refer to a textbook on analysis. But the theorem has a very powerful intuitive meaning which makes it practically obvious. Indeed, draw the curve of the function $y = f(x)$ in a cartesian coordinate system, then draw two vertical lines at $x = a$ and $x = b$ from the abscissa to their intersections with the curve and mark these intersections as points A and B , respectively. The straight line through A and B is the *chord* of the curve through these points. Denote the angle which the chord makes with the abscissa by α , hence $\tan \alpha = (f(b) - f(a))/(b - a)$ by elementary trigonometry. Then displace the chord parallel to itself such that the points A and B move closer together and continue until they merge into one point C , say; at this stage the chord becomes the tangent to $f(x)$ at the point C . The angle which the tangent makes with the abscissa is α by construction, and we know that $\tan \alpha$ is equal to $f'(c)$, where c is the foot of the perpendicular dropped from C onto the abscissa. This proves the theorem in an intuitive way.

For the following it will be convenient to rewrite Eq. (1) by setting $a = x$, $b = x + h$ and $c = a + \theta h$ with $\theta \in [0, 1]$, hence

$$f(x + h) = f(x) + hf'(x + \theta h) \quad (2)$$

2.) A generalization of Lagrange's mean value theorem is the following theorem due to Taylor.

Taylor's Theorem: if the function $f(x)$ is continuous and has continuous derivatives up to the n th in the interval $[x, x + h]$, then

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{h^n}{n!}f^{(n)}(x + \theta h) \quad (3)$$

where $\theta \in [0, 1]$.

¹here the word *differentiable* stands to mean that the derivative dy/dx is also continuous in $[a, b]$.

To make this theorem plausible, consider the function $f(x)$ to be a polynomial of order n . Denoting the polynomial by $P_n(x)$, we have

$$P_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \quad (4)$$

and hence $a_0 = P_n(0)$. Now differentiate $P_n(x)$ once:

$$P'_n(x) = a_1 + 2a_2x + \dots + na_nx^{n-1} \quad (5)$$

and hence $a_1 = P'_n(0)$. For the second derivative we get

$$P''_n(x) = 2a_2 + 3 \cdot 2a_3x + \dots + n(n-1)a_nx^{n-2} \quad (6)$$

and hence $a_2 = (1/2)P''_n(0)$. Continuing the process we get

$$a_3 = \frac{1}{3!}P_n^{(iii)}(0), \quad a_4 = \frac{1}{4!}P_n^{(iv)}(0), \quad \dots, \quad a_n = \frac{1}{n!}P_n^{(n)}(0) \quad (7)$$

and hence, substituting the expressions for the coefficients a_k into Eq. (4), we get

$$P_n(x) = P_n(0) + P'_n(0)x + \frac{1}{2!}P''_n(0)x^2 + \dots + \frac{1}{n!}P_n^{(n)}(0)x^n \quad (8)$$

If we replace in the latter formula x by $x+h$, keep x fixed and consider $P_n(x+h)$ to be a function of h , then we can rewrite Eq. (8) as

$$P_n(x+h) = P_n(x) + P'_n(x)h + \frac{1}{2!}P''_n(x)h^2 + \dots + \frac{1}{n!}P_n^{(n)}(x)h^n \quad (9)$$

Now assume that $f(x)$ is an arbitrary function satisfying the conditions of Taylor's theorem and construct a polynomial $P_n(x)$ such that at a fixed value of x we have

$$P_n(x) = f(x), \quad P'_n(x) = f'(x), \quad P''_n(x) = f''(x), \quad \dots, \quad P_n^{(n)}(x) = f^{(n)}(x) \quad (10)$$

then Taylor's theorem means that the difference between $f(x+h)$ and $P_n(x+h)$ is equal to $f^{(n+1)}(x+\theta h)h^{n+1}/(n+1)!$. This difference must be proportional to h^{n+1} . Indeed, if it depended on a lower power of h , then that term could be absorbed into the polynomial. The precise form of the difference is plausible by analogy with Lagrange's mean value theorem, Eq. (2).

3.) Assume that the function $f(x)$ has continuous derivatives of any order in the interval $[x, x+h]$. Therefore we can choose n in Eq. (3) arbitrarily large. Then, if in the limit of $n \rightarrow \infty$ we have $f^{(n)}(x+\theta h) \rightarrow 0$, the function $f(x)$ can be expanded in a power series:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots + \frac{h^n}{n!}f^n(x) + \dots \quad (11)$$

(Taylor series).

We can give the Taylor series a different form if we denote the fixed point x by a and then set $x = a+h$, hence

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \dots + \frac{1}{n!}f^n(a)(x-a)^n + \dots \quad (12)$$

In this form one says that the function $f(x)$ is expanded about the point $x = a$. In the particular case of $a = 0$ we get the MacLaurin series:

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \dots + \frac{1}{n!}f^{(n)}(0)x^n + \dots \quad (13)$$

4.) Examples:

(i) Let $f(x) = \exp(x)$. Then $f'(x) = f(x)$, $f''(x) = f(x)$, \dots , $f^{(n)}(x) = f(x)$, \dots , and hence we get the MacLaurin series of the exponential function:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + \dots \quad (14)$$

(ii) Similarly we get the MacLaurin series of the sine and cosine of x :

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} \pm \dots \quad (15)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots + (-1)^n \frac{1}{(2n)!}x^{2n} \pm \dots \quad (16)$$

(iii) Next consider the natural logarithm of x : $f(x) = \ln x$. Its first derivative is $f'(x) = 1/x$, and hence $f'(x) \rightarrow \infty$ for $x \rightarrow 0$. In fact, $\ln x$ itself is singular at $x = 0$: $\ln x \rightarrow -\infty$ for $x \rightarrow 0$. Therefore we cannot write down a series expansion about zero for the logarithm. However, we can expand the logarithm about some point $x > 0$, *e.g.* about $x = 1$. Proceeding as in Eq. (12) we get

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots + (-1)^{n+1} \frac{1}{n}(x-1)^n \pm \dots \quad (17)$$

One can show that this series represents the logarithm only in the semi-open interval $(0, 2]$.

To see that the series converges for $x = 2$ we note that we have an alternating series with general term tending to zero for $n \rightarrow \infty$. Therefore we get the following series for the natural logarithm of 2:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + (-1)^{n+1} \frac{1}{n} \pm \dots$$

Note however that this series is of no practical value for the evaluation of the log of 2 because of its very slow convergence. For values of $x > 2$ one can show that the series (17) diverges.

(iv) Consider the expression $1/|\mathbf{a} - \mathbf{r}|$ where \mathbf{a} and \mathbf{r} are two vectors. Expressions of this type occur frequently in applications, for instance to describe the potential of a point charge, located at \mathbf{a} , at the point \mathbf{r} . From vector algebra one has

$$|\mathbf{a} - \mathbf{r}| = \sqrt{a^2 - 2ar\mu + r^2}$$

where $\mu = \cos \alpha$, and α is the angle between the vectors \mathbf{a} and \mathbf{r} . Assume that $r > a$. Then we have

$$\frac{1}{|\mathbf{a} - \mathbf{r}|} = \frac{1}{r} \frac{1}{\sqrt{1 - 2\mu x + x^2}}$$

with $x = a/r$ (and we note that $x < 1$). Thus consider this function as a function of x and find the expansion about $x = 0$. The expansion coefficients depend on μ ; we denote the n th coefficient by $P_n(\mu)$, *i.e.*

$$\frac{1}{|\mathbf{a} - \mathbf{r}|} = \frac{1}{r} (P_0(\mu) + P_1(\mu)x + P_2(\mu)x^2 + \dots)$$

It is left as an exercise to show that

$$P_0(\mu) = 1, \quad P_1(\mu) = \mu, \quad P_2(\mu) = \frac{1}{2}(3\mu^2 - 1), \quad P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu)$$

Thus the expansion coefficients are polynomials in μ , called Legendre polynomials. They have the following general properties:

$$\begin{aligned} P_n(1) &= 1, & P_n(-1) &= (-1)^n \\ P_n(-\mu) &= (-1)^n P_n(\mu) \\ \int_{-1}^1 P_m(\mu)P_n(\mu) d\mu &= \begin{cases} 0 & \text{if } m \neq n \\ 2/(2n+1) & \text{if } m = n \end{cases} \end{aligned}$$

and $P_n(\mu)$ has n distinct zeros in the open interval $(-1, 1)$.