Relativistic Quantum Mechanics

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Contents:

1. Notation and units.

We denote the components of the space-time 4-vector x by $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ or, in compact form,

$$\mathrm{x} = (x^0,\, x^1,\, x^2,\, x^3) \qquad \text{contravariant components}$$

The dual vector has the following covariant components:

$$x_0 = ct, \quad x_1 = -x, \quad x_2 = -y, \quad x_3 = z.$$

The invariant square of x is:

$$\mathrm{x}^2 \equiv \mathrm{x{\cdot}x} = x^\mu x_\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = \text{invariant}$$

where summation over the repeated index μ, one upper and one lower, is implied (*Einstein summation convention*).

We define the metric tensor $g^{\mu\nu}$:

$$g^{\mu\nu} = diag(1, -1, -1, -1)$$

hence

$$x^\mu = g^{\mu\nu} x_\nu, \qquad \mu = 0, 1, 2, 3$$

this is called the "raising of the subscript".

The dual metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = diag(1, -1, -1, -1)$$

hence

$$x_\mu = g_{\mu\nu} x^\nu, \qquad \mu = 0, 1, 2, 3$$

("lowering of the superscript").

Lorentz boost with boost velocity v in x direction:

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad ct' = \gamma(ct - vx/c) \tag{1}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Inverse transformation: change sign of v and exchange the primed and unprimed coordinates.

The energy-momentum 4-vector p is

$$\mathrm{p} = (E/c, \vec{p})$$

and its Lorentz invariant square is

$$\mathrm{p}^2 = p^\mu p_\mu = (E/c)^2 - \vec{p}^2 = (mc)^2 = \text{invariant}$$

We get the rest energy E_0 by setting the momentum equal to nought:

$$E_0 \equiv E_{|\vec{p}=0} = mc^2$$

The relativistic kinetic energy is defined by

$$T = E - mc^2$$

and we can easily check that we get in the nonrelativistic limit $|\vec{p}| \ll mc$ the well known formula $T = p^2/2m$.

Energy and momentum expressed in terms of the particle velocity v are

$$E = \gamma mc^2, \qquad \vec{p} = \gamma m\vec{v}, \qquad \text{hence} \qquad \vec{v} = c\frac{\vec{p}c}{E}.$$

Units:
We use natural units, defined by $c = 1$ and $\hbar = 1$.
To carry out the conversion to GeV units use the conversion factors

$$\hbar c = 197.327\,\text{MeV fm}, \qquad (\hbar c)^2 = 0.3894\,\text{GeV}^2\ \text{mbarn}$$

where 1 fm $= 10^{-15}$ m (femto-meter or Fermi), and 1 mbarn $= 10^{-31}$ m^2 (milli-barn).
For numerical estimates use $\hbar c = 200$ MeV fm $= 200$ eV nm, $(\hbar c)^2 = 0.4\,\text{GeV}^2$ mbarn.

Typical particle masses (approximate values):

Particle	symbol	Mass in MeV
Electron	e^-	0.5
Muon	μ^-	106
Charged pion	$\pi^\pm$	140
Neutral pion	π^0	135
Charged kaon	$K^\pm$	494
Neutral kaon	K^0	498
Proton	p	938
Neutron	n	940

2. Relativistic kinematics.

For a collision of particles a and b, which gives rise to the creation of particles c, d, ..., we write the reaction equation

$$a + b \to c + d + \ldots$$

Particles a and b are the *incident* particles, particles c, d, etc. are the *outgoing* particles.
The four-momenta of particles a and b will be denoted by p_1 and p_2, respectively, and those of the outgoing particles by p_3, p_4 etc. Their masses are given by $\mathrm{p}_i^2 = m_i^2$.
The kinematics of collision processes is described in various reference frames of which we discuss here the *laboratory frame* (LAB) and the *centre-of mass frame* (CMS) in some detail.

LAB frame: particle a is the *incident* particle, particle b is the *target* particle. Their four-momenta are

$$\mathrm{p}_1 = (E_{lab}, 0, 0, p_{lab}), \qquad \mathrm{p}_2 = (m_2, 0, 0, 0) \tag{2}$$

The square of the total four-momentum of the system is

$$s = (\mathrm{p}_1 + \mathrm{p}_2)^2 = m_1^2 + m_2^2 + 2m_2 E_{lab} \tag{3}$$

Solving for p_{lab} we get

$$p_{lab} = \sqrt{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2]}/2m_2 \tag{4}$$

<u>CMS frame:</u>
by definition of the CMS, the incident particles have 3-momenta of equal magnitude and opposite direction:

$$\mathrm{p}_1^{cms} = (E_1^*, 0, 0, p^*), \qquad \mathrm{p}_2^{cms} = (E_2^*, 0, 0, -p^*). \tag{5}$$

hence

$$s = (E_1^* + E_2^*)^2. \tag{6}$$

Thus $\sqrt{s}$ is the total CMS energy of the system. Solving for p^* and comparing with Eq. (4) we get

$$p^* = p_{lab}\frac{m_2}{\sqrt{s}} \tag{7}$$

In particle colliders, the masses of the beam particles are negligible in comparison with their momenta. Assuming the collider to be symmetrical, such as usually in pp, $\bar{p}p$ or e^+e^- colliders, we have for the total CMS energy

$$E_{cm} \equiv \sqrt{s} = 2E_1 = 2E_2$$

and the entire energy of the two beam particles is available for particle reactions. This is unlike the LAB (or fixed target) case, where the LAB kinetic energy of the system is lost to reactions. For example, the equivalent LAB energy of electrons incident on electrons at rest, that corresponds to a CMS energy of 90 GeV (*i.e.* the energy needed to produce a Z boson), is $E_{lab} = s/2m_e \approx 8 \times 10^6$ GeV, *i.e.* many orders of magnitude greater than the highest energies to which electrons have been accelerated to date.

3. The Schrödinger equation

Fundamental for quantum mechanics is the concept of particle-wave duality, which has a firm base in a large number of experiments. Historically it was the particle-wave duality of light which was recognised first by Einstein, and later de Broglie theoretically and independently Davisson and Germer experimentally established the particle-wave duality for electrons. Formally particle-wave duality is expressed by the Einstein-de Broglie relations:

$$E = \hbar\omega, \qquad p = \hbar k, \tag{8}$$

which relate the particle characteristics E (energy) and p (momentum) to the wave characteristics ω (frequency) and k (wave number). Here $\hbar = h/2\pi$, where h is Planck's constant.

Implied in the statement of particle-wave duality is also that the particle must be describable in terms of a wave function. In the simplest case of a free particle travelling in the x direction this is the plane wave

$$\psi(x, t) = e^{i(kx - \omega t)} \tag{9}$$

Moreover, we demand that the wave function contains the complete information on the state of motion of the particle at time t. This implies that the wave function also contains the information needed to predict the state of motion of the particle at later times or, as one says, that the wave function determines the *time evolution* of the particle's state of motion, *i.e.* we must have an equation of the form

$$\frac{\partial\psi(x,t)}{\partial t} = \hat{O}\psi(x,t)$$

where $\hat{O}$ is some operator which we must now identify.

If we substitute the wave function (9) and use the Einstein-de Broglie relations (8), we get on the left-hand side

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = E\psi(x,t) \tag{10}$$

In the nonrelativistic case we identify the energy of the free particle with its kinetic energy, *i.e.* using once more the de Broglie relation we have

$$E = \frac{p^2}{2m} = \frac{\hbar^2}{2m}k^2$$

But for the plane wave we also have the identity

$$k^2\psi(x,t) = -\frac{\partial^2\psi(x,t)}{\partial x^2}$$

and therefore finally

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$$

We have thus identified the operator $\hat{O}$: up to a factor $i\hbar$ it is the kinetic energy operator which we denote by $\hat{T}$:

$$\hat{T} = \frac{\hat{p}^2}{2m} \qquad \text{where } \hat{p} \text{ is the momentum operator:} \qquad \hat{p} = -i\hbar\frac{\partial}{\partial x}$$

The wave equation of the free particle therefore takes on the form:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \hat{T}\psi(x,t)$$

The decisive step made by Schrödinger in the development of quantum mechanics was the realization that the correct form of the wave equation for an electron in a potential $V(x)$ is

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \hat{H}\psi(x,t) \tag{11}$$

where

$$\hat{H} = \hat{T} + V(x) \tag{12}$$

is the Hamiltonian operator, which therefore appears as a direct generalization of the Hamilton function of classical mechanics.

The generalization to three dimensions is straight forward: the wave function becomes a function of $\vec{r} = (x, y, z)$,

$$\psi = \psi(\vec{r}, t)$$

and the momentum operator is a vector differential operator with three components,

$$\hat{p} = -i\hbar\nabla = -i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

The Schrödinger equation takes the same form as before, except that the potential energy now also depends on the three spatial coordinates:

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = \hat{H}\psi(\vec{r},t) = \left(\hat{T} + V(\vec{r})\right)\psi(\vec{r},t) \tag{13}$$

Probability interpretation of the wave function

Of fundamental importance for quantum mechanics is the physical interpretation of the wave function. This is radically different from the interpretation in the case of any classical waves in elastic media, where the wave function is a displacement of a body, such as a string or a membrane.

The generally accepted physical interpretation of the wave function in quantum mechanics, which was first proposed by Max Born, is that

$$|\psi(\vec{r},t)|^2 d^3r$$

is the probability of finding the electron at time t in the volume element d^3r at the point $\vec{r}$. An immediate implication of this interpretation is that the integral of $|\psi(\vec{r},t)|$ over all space is expressing certainty, *i.e.*

$$\int_{\text{all space}} |\psi(\vec{r},t)|^2 d^3r = 1 \tag{14}$$

Mathematically this is the expression of *square integrability* and of *normalizability* of the wave function. Important is that the wave function under the integral is time dependent whereas the right-hand side is constant. This is the statement of *conservation of probability.*

Now, the modulus-squared of the wave function is the product of the wave function and its complex conjugate. Therefore, to give the above statements a physical content we must also demand that the complex conjugate wave function is also the solution of a wave equation. This wave equation is obtained from Eq. (13) by taking the complex conjugate:

$$-i\hbar\frac{\partial\psi^*(\vec{r},t)}{\partial t} = (\hat{H}\psi(\vec{r},t))^* \tag{15}$$

If we multiply Eq. (13) by ψ^* and Eq. (15) by ψ and subtract we get:

$$i\hbar\frac{\partial(\psi^*\psi)}{\partial t} = \psi^*\left(\hat{H}\psi\right) - \psi\left(\hat{H}\psi\right)^* \tag{16}$$

and if we integrate this equation over all space and change on the left-hand side the order of integration and differentiation w.r.t. time, then we get, using Eq. (14),

$$\int_{\text{all space}} d^3r \left[\psi^* \left(\hat{H}\psi\right) - \psi \left(\hat{H}\psi\right)^*\right] = 0$$

which means that the Hamiltonian $\hat{H}$ is hermitian, with the immediate implication that its eigenvalues are real and its eigenfunctions are orthogonal.

We can also verify that Eq. (16) is of the form of a *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\jmath} = 0 \tag{17}$$

where

$$\rho = \psi^* \psi \qquad \text{and} \qquad \vec{\jmath} = \frac{\hbar}{2im} \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

are the *probability density* and the *probability current density*, respectively.

Integrating over a finite volume and using Stokes' theorem we get

$$\frac{d}{dt} \int_V \rho(\vec{r}, t) d^3r = -\oint_S \vec{\jmath} \cdot d\vec{S} \tag{18}$$

which means that the increase of the probability to find the particle in the volume V is equal to the inward flux of probability through the surface S enclosing V.

Superposition principle

The Schrödinger equation (13) is linear and homogeneous in the wave function. It follows that, if ψ_1 and ψ_2 are two solutions of the Schrödinger equation, then their linear superposition

$$\psi = c_1 \psi_1 + c_2 \psi_2$$

is also a solution; here c_1 and c_2 are arbitrary complex coefficients. This is the formal expression of the *superposition principle*, which is a fundamental principle of quantum mechanics, as especially emphasised by Dirac.

Eigenvalues and eigenfunctions.

If the Hamiltonian H has no explicit time dependence, then we can represent the wave function as a product $\psi(x, t) = u(x)\, f(t)$, hence, after substitution into Eq. (13),

$$i\hbar\, u(x) \frac{df(t)}{dt} = H u(x) f(t)$$

or after division by $u(x) f(t)$

$$i\hbar \frac{1}{f(t)} \frac{df(t)}{dt} = \frac{1}{u(x)} H u(x)$$

The left-hand side is a function of t and the right-hand side is a function of x. For the two to be identical they have to be equal to a constant, which we denote by E. Thus we get two equations:

$$i\hbar \frac{df(t)}{dt} = E\, f(t) \tag{19}$$

$$H u(x) = E u(x) \tag{20}$$

Equation (20) is immediately solved: we get

$$f(t) = \exp(-iEt/\hbar)$$

and hence $\psi(x,t) = u(x)e^{-iEt/\hbar}$, *i.e.*

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = E\,\psi(x,t)$$

Comparison with Eq. (10) shows that E is the energy, only now it is the total energy of the particle in the potential V, whereas previously it was the energy of the free particle.

The second equation, Eq. (20), is an eigenvalue equation: in general it has solutions only for some values of E, called the *eigenvalues* of H. The corresponding solutions $u(x)$ are the *eigenfunctions* of H. We must distinguish two classes of solutions: *bound states* and *continuous states*. Bound states arise when the potential energy has a minimum. Consider the case of the Coulomb potential: $V(r) = -\gamma/r$; thus $V(r)_{|r\to\infty} \to 0$. Then for $E < 0$ there are bound states, which can be shown to a discrete set of eigenvalues E_α. For $E > 0$ the particle can escape to infinity. In this case bound states do not exist and there is a continuum of eigenvalues. The complete set of eigenvalues is the *spectrum* of H.

Important for the interpretation of E as the energy is that it is real. The reality of E is guaranteed by the Hermiticity of H. For the mathematical proof of this statement consult a textbook on quantum mechanics. Generally one can show that a Hermitian operator has only real eigenvalues. The converse statement also holds: if a linear operator has only real eigenvalues, then it is Hermitian.

To solve the eigenvalue equation for H we need to know its explicit form. However, one can make some general statements concerning the eigenfunctions $u_\alpha(x)$:

- eigenfunctions belonging to different eigenvalues are orthogonal;
- the eigenfunctions of H form a *complete* set.

Orthogonality of $\psi_\alpha(x,t) = u_\alpha(x,t)e^{-iET/\hbar}$ is expressed by

$$\int \psi_\alpha^* \psi_\beta dx = 0 \quad \text{if} \quad \beta \neq \alpha$$

where the integral is taken over all space. If we combine this statement with the normalization condition, Eq. (14), then

$$\int \psi_\alpha^* \psi_\beta dx = \delta_{\alpha\beta} \tag{21}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol:

$$\delta_{\alpha\beta} = \begin{cases} 1 \text{ if } \beta = \alpha \\ 0 \text{ if } \beta \neq \alpha \end{cases}$$

Completeness of the set of eigenfunctions ψ_α of H means that any solution of the Schrödinger equation (13) can be represented as a linear superposition of ψ_α:

$$\psi(x,t) = \sum_\alpha c_\alpha \psi_\alpha(x,t)$$

and it is understood that if all or part of the energy spectrum is continuous, then the summation over α has to replaced by integration.

4. Relativistic wave equation (Klein-Gordon equation); antiparticles.

To construct a relativistic wave equation we shall use the relations that provide the transition from classical mechanics to quantum mechanics:

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad p_x \to -i\hbar\frac{\partial}{\partial x}, \quad p_y \to -i\hbar\frac{\partial}{\partial y}, \quad p_z \to -i\hbar\frac{\partial}{\partial z}, \tag{22}$$

together with the relativistic energy-momentum relation

$$E^2 = (pc)^2 + m^2c^4$$

which immediately gives the Klein-Gordon equation:

$$-\hbar^2\partial_t^2\phi(\mathrm{x}) = \left[-(\hbar c)^2\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) + m^2c^4\right]\phi(\mathrm{x}) \tag{23}$$

where we have used the shorthand form of the differential operators:

$$\partial_t \equiv \frac{\partial}{\partial t}, \qquad \partial_x \equiv \frac{\partial}{\partial x}, \quad \partial_y \equiv \frac{\partial}{\partial y}, \quad \partial_z \equiv \frac{\partial}{\partial z}.$$

We can rewrite the Klein-Gordon equation in a relativistically symmetric form which clearly exhibits its relativistic invariance. To do this we define the four-dimensional generalization of the momentum operator:

$$\hat{\mathrm{p}} = \left(\hat{p}^0, \hat{p}^1, \hat{p}^2, \hat{p}^3\right) = \left(i\hbar\partial^0, i\hbar\partial^1, i\hbar\partial^2, i\hbar\partial^3\right) = \left(i\frac{\hbar}{c}\partial_t, -i\hbar\partial_x, -i\hbar\partial_y, -i\hbar\partial_z\right)$$

and hence the four-dimensional generalization of the Laplacian operator:

$$\partial_\mu\partial^\mu = \frac{1}{c^2}\partial_t^2 - \nabla^2$$

With these definitions the Klein-Gordon equation takes on the manifestly invariant form

$$\left[\partial_\mu\partial^\mu + \left(\frac{mc}{\hbar}\right)^2\right]\phi(\mathrm{x}) = 0 \tag{24}$$

provided that the wave function $\phi(\mathrm{x})$ is a Lorentz scalar or pseudoscalar.

This latter point appears here as a mathematical requirement, but it should be stressed that it has an important physical implication: we can apply the Klein-Gordon equation only to particles which are described by *scalar* or *pseudoscalar* wave functions. Such particles do exist, for instance pions and kaons, which are pseudoscalar particles, but electrons are *not* scalar particles and their wave functions are correspondingly not scalars.

Note that $mc/\hbar$ is the inverse of the Compton wavelength of the particle of mass m.

To derive a continuity equation we write down the Klein-Gordon equation for the complex conjugate wave function $\phi^*(\mathrm{x})$, multiply it by $\phi(\mathrm{x})$, multiply Eq. (23) by $\phi^*(\mathrm{x})$ and subtract the resulting equations. As a result we get

$$\partial_t \rho + \nabla \cdot \vec{j} = 0 \tag{25}$$

where

$$\rho(\mathrm{x}) = i\left(\phi^* \partial_t \phi - \phi \partial_t \phi^*\right), \qquad \vec{j} = -i\left(\phi^* \nabla \phi - \phi \nabla \phi^*\right)$$

Important is that $\rho(\mathrm{x})$ can be negative as well as positive, so it is not a probability density.

In the particular case of a plane wave, $\phi(\mathrm{x}) = N \exp(i(\vec{p}\cdot\vec{r} - Et)/\hbar)$, we have

$$\rho = 2E|N|^2, \qquad \vec{j} = 2\vec{p}|N|^2$$

where

$$E = \pm\sqrt{(pc)^2 + m^2c^4}$$

Thus the negative values of ρ are related to negative energies. These negative energy solutions cannot be dropped as unphysical because the remaining positive energy wave functions would no longer form a complete set. Therefore a physical explanation for these states is needed. This was given independently by Feynman and by Stückelberg.
As before, in the nonrelativistic case, we can get the equivalent integral form (18) of the continuity equation:

$$\frac{d}{dt}\int_V \rho(\vec{r},t) d^3r = -\oint_S \vec{j}\cdot d\vec{S} \tag{26}$$

and with the usual requirement that the fields vanish at large distances sufficiently fast for the surface integral to vanish when the integral is taken over all space we get

$$\frac{d}{dt}\int_{\text{all space}} \rho(\vec{r},t) d^3r = 0 \tag{27}$$

and hence $\int_{\text{all space}} \rho(\vec{r},t) d^3r = \text{constant}$. This means in particular that the sign of the integral of ρ over all space is conserved. Therefore, if ρ is multiplied by the charge of the particle, then it can be interpreted as the *charge density*; similarly we must also multiply $\vec{j}$ by the particle's charge giving us the *current density* (Pauli-Weisskopf interpretation).

Next we note that $\rho(\vec{r},t)$ transforms like time since the wave function $\phi(\mathrm{x})$ is a scalar and the operator $\partial/\partial t$ transforms like time. It is therefore natural to combine ρ and $\vec{j}$ into a four-vector:

$$j^\mu = (\rho, \vec{j}) = -ie\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right) \tag{28}$$

where we have included the charge factor $-e$. In terms of this four-vector current density the continuity equation (25) takes on the form

$$\partial^\mu j_\mu = 0 \tag{29}$$

Now let us streamline our notation by using from now on units such that $\hbar = 1$ and $c = 1$. In these units masses, energies and momenta all have the same dimension of energy, and length has the dimension of inverse energy. Thus the scalar product p·x of four-momentum and the space-time four-vector is dimensionless.

In these units the plane wave is of the form

$$\phi(\mathrm{x}) = Ne^{-i\mathrm{p}\cdot\mathrm{x}} \tag{30}$$

and the current density is

$$j^\mu(e^-) = -2e|N|^2(E,\vec{p}) \tag{31}$$

where the argument (e^-) indicates that this is for a *negatively* charged particle. If we now change the sign of the charge we get

$$j^\mu(e^+) = +2e|N|^2(E,\vec{p}) = -2e|N|^2(-E,-\vec{p}) \tag{32}$$

i.e. the particle with *positive* charge travelling with energy E and momentum $\vec{p}$ is the same as a particle with *negative* charge and with energy $(-E)$ and momentum $(-\vec{p})$. Both particles have the same mass m because the currents are constructed from wave functions satisfying the same wave equation. Such a pair of particles, having equal mass but opposite charges, are called the *antiparticles* of each other: one particle is arbitrarily designated the particle, and the other one is then the antiparticle.

Another useful way of seeing the relationship between particle and antiparticle is based on a consideration of the time-dependent factor of the wave function: if we have a particle of negative energy E, then we can write this factor as

$$e^{-iEt} = e^{-i(-E)(-t)} \tag{33}$$

where the expression on the right-hand side corresponds to a particle of *positive* energy $(-E)$ travelling *backwards in time*! Together with the previous discussion we can therefore conclude that

> an antiparticle travelling forward in time is identical with a particle travelling backward in time.

This is the Feynman-Stückelberg interpretation of the negative energy states. Later on we shall see that the same interpretation applies also in the case of Dirac's relativistic equation of the electron.

5. Electromagnetic interactions of spin-0 particles.

5.1 Perturbation theory; Fermi's Golden Rule.

We are going to study collision processes in which the physical state of a system after the collision differs from that before the collision. Problems of this kind require time dependent perturbation theory. TDPT can be applied in cases in which the Hamiltonian is explicitly time dependent but also in cases with explicitly time independent Hamiltonians, if the interaction that causes the transition from one state into another persists for only a finite duration of time. Such is the case in collisions on short range potentials, if the particle is initially at a large distance from the region of nonzero potential, and after passing through that region is again travelling as a free particle, but in a different state than before the interaction.

Thus we consider the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi, \qquad \text{with} \quad H = H_0 + V(t)$$

where it is assumed that the eigenfunctions ϕ_n and eigenvalues E_n of H_0 are known, and $V(t)$ is a small perturbation. Expanding ψ into the complete set of eigenfunctions ϕ_n,

$$\psi = \sum_n a_n(t)\phi_n e^{-iE_n t/\hbar}$$

we get the equivalent form of the Schrödinger equation in E-representation:

$$i\hbar\frac{da_m(t)}{dt} = \sum_n a_n(t)V_{mn}e^{-i(E_n-E_m)t/\hbar}$$

where $V_{mn} = (\phi_m, V(t)\phi_n)$ is the matrix element of $V(t)$ between the mth and nth eigenstates of H_0.
Assuming that up to the time $-T/2$ the system was in the state ϕ_i, we have the initial conditions

$$a_i(-T/2) = 1, \qquad a_n(-T/2) = 0 \text{ for } n \neq i$$

which for some final state ϕ_f gives the approximate solution for times $t \geq T/2$:

$$a_f(t) = -i\int_{-T/2}^{T/2} V_{fi}(t')e^{-i(E_i-E_f)t'/\hbar}\,dt'$$

(it is assumed that the perturbation $V(t)$ is switched off at $t = T/2$).

In the particular case of a potential independent of time we get, in the limit of $T \to \infty$,

$$a_f = -2\pi i V_{fi}\delta(E_i - E_f)$$

where the Dirac delta-function ensures energy conservation (this is, of course, meaningful only if the states ϕ_n are degenerate, since otherwise the system would persist in the initial state; for a more detailed discussion read, e.g., D.I. Blokhintsev, *Osnovy kvantovoi mekhaniki*, 3rd edition, 1961, Chapter XIV).

The transition probability from ϕ_i to ϕ_f is

$$\begin{aligned} w_\infty \equiv |a_f|^2 &= 4\pi^2|V_{fi}|^2\,[\delta(E_i - E_f)]^2 \\ &= 4\pi^2|V_{fi}|^2\delta(E_i - E_f)\left[\lim_{T\to\infty}\frac{1}{2\pi\hbar}\int_{-T/2}^{T/2} e^{i(E_i-E_f)t/\hbar}\,dt\right] \\ &= \frac{2\pi}{\hbar}|V_{fi}|^2\delta(E_i - E_f)\lim_{T\to\infty}[T] \end{aligned}$$

and the transition probability per second is

$$w_{fi} = \lim_{T\to\infty}\frac{w_\infty}{T} = \frac{2\pi}{\hbar}|V_{fi}|^2\delta(E_i - E_f)$$

The usual situation in particle collisions is that one measures the transition rate into a group of closely spaced final states. Denoting the density of final states by $\rho(E_f)$, then $\rho(E_f)dE_f$ is the number of final states in the interval $(E_f, E_f + dE_f)$. Multiplying w_{fi} by this number and integrating over E_f we get, on account of the δ function, *Fermi's Golden Rule*:

$$W_{fi} = \frac{2\pi}{\hbar}|V_{fi}|^2\rho(E_i) \tag{34}$$

Examples:
(i) Interaction of spinless charged particles with electromagnetic field.
Assume the oscillating e.m. field to have frequency ω, then $V(t) \approx \exp(-i\omega t)$, and the transition amplitude is

$$a_f \approx \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{-i(E_i - E_f + \omega)t}\, dt \approx \delta(E_f - E_i - \omega)$$

i.e. $E_f = E_i + \omega$, which means that the particle has absorbed the energy ω from the e.m. field.
(ii) Interaction of spinless charged *anti*particle with electromagnetic field.
According to the Feynman interpretation, the incident (outgoing) antiparticle of *positive* energy E_i (E_f) is equivalent to a particle of *negative* energy $-E_i$ $(-E_f)$ travelling backwards in time. With $V(t) \approx \exp(-i\omega t)$, as before, we get therefore

$$a_f \approx \lim_{T\to\infty} \int_{-T/2}^{T/2} \left(e^{-i(-E_i)t}\right)^* e^{-i\omega t} e^{-i(-E_f)t}\, dt \approx \delta(E_f - E_i - \omega)$$

and hence again $E_f = E_i + \omega$, in other words the antiparticle has also absorbed the energy ω from the field, as expected. This shows that the Feynman interpretation is consistent with our physical expectation in the case of interactions.

We can similarly check that we get the intuitively expected results in the cases of particle-antiparticle pair creation and annihilation.

5.2 Elastic scattering of spin-0 particles.
Consider the collision of two unlike spin-0 particles. We shall call them π^+ and K^+. We will treat them as point-like particles, which is deficient in so far as the real mesons have a size of the order of one Fermi. The correct description takes this into account by assigning *form factors* to the mesons.

The pion is described by the Klein-Gordon equation,

$$\left(\partial^\mu \partial_\mu + m^2\right)\Phi(x) = 0$$

The interaction with electromagnetic field is taken into account by the replacement of the derivative operator by the covariant derivative

$$i\partial^\mu \to i\partial^\mu - eA^\mu$$

where A^μ is the electromagnetic four-vector potential. Thus

$$\left[(\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu) + m^2\right]\Phi(x) = \left[+\partial^\mu\partial_\mu + m^2 + ie\left(\partial^\mu A_\mu + A^\mu \partial_\mu\right) - e^2 A^\mu A_\mu\right]\Phi(x) = 0 \tag{35}$$

We neglect the term proportional to e^2 as being of second order of smallness. Then the KG equation takes the form

$$\left(\partial^\mu \partial_\mu + m^2\right)\Phi(x) = -V(x)\Phi(x)$$

with

$$V(x) = ie\left(\partial^\mu A_\mu + A^\mu \partial_\mu\right)$$

The sign of the potential V was chosen such as to give the correct form of the Schrödinger equation in the nonrelativistic limit.

Having identified the interaction energy, we can now use the result of first-order perturbation theory, Eq. (34), and write down the transition amplitude:

$$\begin{aligned} T_{fi} &= -i\int d^4x\, \Phi_f^* V \Phi_i \\ &= e\int d^4x\, \Phi_f^*\left(\partial^\mu A_\mu + A^\mu \partial_\mu\right)\Phi_i \end{aligned} \tag{36}$$

The first term in brackets is transformed by integration by parts:

$$\int d^4x\, \Phi^* \partial^\mu \left(A_\mu \Phi_i\right) = -\int d^4x\, \left(\partial^\mu \Phi_f\right)^* A_\mu \Phi_i$$

where we have dropped the surface term. Thus the transition amplitude takes on the form

$$T_{fi} = e\int d^4x\, \left[\Phi_f^*\left(\partial_\mu\ \Phi_i\right) - \left(\partial_\mu \Phi_f^*\right)\Phi_i\right] A^\mu$$

The expression in brackets has the form of a current (cf. Eq. (25)), but involving different wave functions Φ_i and Φ_f. It is therefore a *transition* current:

$$j^\mu(\pi^+) = ie\left[\Phi_f^*\left(\partial^\mu \Phi_i\right) - \left(\partial^\mu \Phi_f^*\right)\Phi_i\right]$$

and we can write the transition amplitude in the form of

$$T_{fi} = -i\int d^4x\, j_\mu(\pi^+) A^\mu$$

The particle in the initial and final states can be represented by plane waves:

$$\Phi_i = N_i e^{-ip_i x}, \qquad \Phi_f = N_f e^{-ip_f x}$$

where N_i and N_f are normalization factors, hence

$$j^\mu(\pi^+) = eN_i N_f \left(p_i + p_f\right)^\mu e^{i(p_f - p_i)x} \tag{37}$$

The formalism so far is appropriate for the interaction of the π^+ particle with an arbitrary e.m. field A^μ. In order to apply it to the reaction $\pi^+ K^+ \to \pi^+ K^+$ we must consider the e.m. field to be emitted by the kaon. Thus we write down the wave equation for A^μ with a source term:

$$\partial^\nu \partial_\nu A^\mu = j^\mu(K^+)$$

where $j^\mu(K^+)$ is the kaon current. Changing our notation, we assign 4-momenta p_1 and p_2 to the incident pion and kaon, respectively, and similarly p_3 and p_4 to the outgoing particles. We have therefore

$$j^\mu(K^+) = eN_2 N_4 \left(p_2 + p_4\right)^\mu e^{iqx}$$

where $q = p_4 - p_2$, and hence

$$A^\mu = (\partial^\nu \partial_\nu)^{-1} j^\mu(K^+) = -\frac{1}{q^2} j^\mu(K^+)$$

The transition amplitude takes now the form of

$$T_{fi} = -i \int d^4x \, j_\mu(\pi^+) \left(-\frac{1}{q^2} \right) j^\mu(K^+)$$

and the integration can be done trivially, giving a four-dimensional delta function with the obvious meaning of overall 4-momentum conservation:

$$T_{fi} = -iN_1N_2N_3N_4 \, (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\mathcal{M}$$

where the *transition matrix* $\mathcal{M}$ is defined by

$$-i\mathcal{M} = (-ie)(p_1 + p_3)^\mu \left(-i\frac{g_{\mu\nu}}{q^2} \right) (-ie)(p_2 + p_4)^\nu$$

Writing the transition matrix in this form, we have identified the following factors:
(i) $(-ie)(p_1 + p_3)^\mu$ is associated with the interaction vertex of the pion with the photon,
(ii) $(-ie)(p_2 + p_4)^\nu$ is associated with the interaction vertex of the kaon with the photon,
(iii) $(-ig_{\mu\nu}/q^2)$ is associated with the exchanged photon. This is called the *photon propagator*. The denominator q^2 is the square of the photon's 4-momentum. For a real photon this would be zero, but in the present case we can check that

$$q^2 = (p_1 - p_3)^2 < 0$$

except at zero scattering angle, where $q^2 = 0$. This can be seen most conveniently in the CMS, where

$$q^2 = -2p^2(1 - \cos\theta)$$

(p is the CMS momentum and θ is the CMS scattering angle). Therefore the photon is *virtual* or *off mass-shell.*

In deriving the result for the scattering matrix, we have arbitrarily considered the kaon to be the source of the e.m. field, absorbed by the pion. The form of our result suggests, and an explicit calculation confirms it, that this distinction is unimportant: the pion and kaon play completely symmetric roles in the reaction.

6.) Calculation of the cross section; phase space; Mandelstam variables.

Elastic π^+K^+ scattering is a particular case of a $2 \to 2$ collision. More generally, any reaction of the type

$$a + b \to c + d$$

belongs to this class of processes. The differential cross section $d\sigma$ is related to the scattering amplitude $\mathcal{M}$ by

$$d\sigma = \frac{1}{F}|\mathcal{M}|^2 dQ$$

where dQ is the Lorentz invariant phase space factor,

$$dQ = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} \tag{38}$$

and F is the flux of incident particles, defined by $F = 2E_1\, 2E_2\, |\vec{v}_1 - \vec{v}_2|$, where $\vec{v}_1$ and $\vec{v}_2$ are the velocities of the incident particles 1 and 2, respectively. The invariant form of the flux factor is

$$F = 4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}$$

Useful are the expressions of F in the LAB and CMS frames:

$$F_{lab} = 4 p_{lab} m_2; \qquad F_{cms} = 4 p_i \sqrt{s}$$

Because of Lorentz invariance, the three expressions of F are equal.

The phase space factor must be further simplified in order to remove the δ function. Three of the six integrations are done trivially, for instance over $d^3 p_4$, removing the three-dimensional δ function $\delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$. This leaves us with an expression for $d\sigma$, in which momentum conservation is explicitly imposed. For the remaining integrations we work in polar coordinates:

$$d^3 p_3 = p_3^2\, dp_3\, d\Omega$$

where the element of solid angle $d\Omega$ is given by $d\Omega = \sin\theta\, d\theta\, d\varphi$, with polar angle θ and azimuth φ. Then, from

$$E_3^2 = p_3^3 + m_3^2$$

we have

$$p_3\, dp_3 = E_3\, dE_3$$

hence

$$dQ = \frac{1}{(4\pi)^2} \delta(E_1 + E_2 - E_3 - E_4) \frac{p_3\, dE_3}{E_4}\, d\Omega$$

To do the integration over energy, it is convenient to define the total initial and final energies,

$$W_i = E_1 + E_2, \qquad W_f = E_3 + E_4$$

hence

$$dW_f = dE_3 + dE_4 = \frac{p_3\, dE_3}{E_3} + \frac{p_4\, dE_4}{E_4}$$

A convenient trick is to continue the calculation in the CMS where, on account of momentum conservation, we have $p_3 = p_4 = p_f$, hence $E_4\, dE_4 = E_3\, dE_3 = p_f\, dp_f$, and therefore

$$dW_f = \frac{W_f}{E_4}\, dE_3$$

which gives

$$dQ_{cms} = \frac{1}{(4\pi)^2}\, \delta(W_i - W_f)\, \frac{p_f\, dW_f}{W_f}\, d\Omega$$

Carrying out the last integration over W_f we impose energy conservation, after which we have in the CMS $W_f = W_i = \sqrt{s}$, *i.e.*

$$dQ_{cms} = \frac{1}{(4\pi)^2} \frac{p_f}{\sqrt{s}} \, d\Omega$$

This completes the phase space calculation for the quasi-elastic collision process. Collecting all terms we get therefore the differential cross section in the CMS

$$d\sigma = \frac{1}{64\pi^2 s} \frac{p_f}{p_i} |\mathcal{M}|^2 \, d\Omega$$

where p_i is the CMS momentum of the incident particles. In the particular case of elastic scattering this formula simplifies on account of $p_f = p_i$.

Mandelstam variables.
The remaining calculation is the evaluation of $\mathcal{M}$ which we do in invariant form. To this end we define the Mandelstam variables s, t and u:

$$\begin{aligned}
s &= (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2 \\
&= (p_3 + p_4)^2 = m_3^2 + m_4^2 + 2p_3 \cdot p_4 \\
t &= (p_1 - p_3)^2 = m_1^2 + m_3^2 - 2p_1 \cdot p_3 \\
&= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2 \cdot p_4 \\
u &= (p_1 - p_4)^2 = m_1^2 + m_4^2 - 2p_1 \cdot p_4 \\
&= (p_2 - p_3)^2 = m_2^2 + m_3^2 - 2p_2 \cdot p_3
\end{aligned} \tag{39}$$

The three Mandelstam variables are not independent: 4-momentum conservation gives

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

but it is frequently convenient to use all three variables. In keeping with the subject of high energy physics, we shall frequently consider the ultrarelativistic case where $s + t + u = 0$.

Carrying out the contraction over the Lorentz indices in the expression of the matrix element $\mathcal{M}$ we get

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} \left(p_1 \cdot p_2 + p_1 \cdot p_4 + p_2 \cdot p_3 + p_3 \cdot p_4\right)^2$$

which in the case of elastic scattering is

$$|\mathcal{M}|^2 = \frac{e^4}{t^2} \left[\left(s - m_1^2 - m_2^2\right) + \left(m_1^2 + m_2^2 - u\right)\right]^2 = \frac{e^4}{t^2}(s-u)^2$$

and in the ultrarelativistic case

$$|\mathcal{M}|^2 = e^4 \left(\frac{s-u}{s+u}\right)^2$$

Substituting this into the expression for the elastic differential cross section we get finally

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \left(\frac{s-u}{s+u}\right)^2$$

where we have also expressed the charge in terms of the fine structure constant: $\alpha = e^2/4\pi$.

Experimental data on elastic scattering are usually expressed in terms of the scattering angle, rather than the Mandelstam variables. Denoting the CMS scattering angle by θ, we have in the CMS the 4-momenta $p_1 = (E, 0, 0, E)$, $p_2 = (E, 0, 0, -E)$ and $p_4 = (E, p_{4x}, p_{4y}, p_{4z})$ with $p_{4z} = -E\cos\theta$, and hence from the above definition we have $u = -2E^2(1 + \cos\theta)$ and with $s = 4E^2$ we get the differential cross section in the form of

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(\frac{3 + \cos\theta}{1 - \cos\theta}\right)^2 \tag{40}$$

We see that the differential cross section has a singularity at $\theta = 0$. This is a characteristic feature of photon exchange or, more generally, of the exchange of a zero-mass particle. This result is well known from Rutherford scattering. Because of this singularity we cannot get a total elastic cross section. However, we can get a meaningful quantity if we integrate over θ from some small angle θ_0 to $\theta = \pi$. This corresponds to the usual experimental procedure, which does not allow detection of particles scattered through very small angles. Thus we get

$$\sigma(\theta \geq \theta_0) = \frac{\pi\alpha^2}{2s}\left(1 + \cos\theta_0 + 8\frac{1 + \cos\theta_0}{1 - \cos\theta_0} + 16\ln\sin\frac{\theta_0}{2}\right)$$

To convert our results to ordinary units we must multiply the cross section formulæ by $(\hbar c)^2 \approx 0.4\,\mathrm{GeV}^2\,\mathrm{mbarn}$.

7.) The Dirac equation.

7.1 Derivation.

In constructing the relativistic quantum mechanical equation of the electron we demand, following Dirac,[1] that the equation be of the form of the time-evolution equation of quantum mechanics,

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi, \tag{41}$$

and that it be Lorentz covariant. Then, since the derivative w.r.t. time is of first order, the Hamiltonian $\hat{H}$ must also linearly depend on the derivatives w.r.t. the coordinates. The only other quantities, on which $\hat{H}$ can depend, are the fundamental constants c, $\hbar$ and m, the electron mass. Thus, on dimensional grounds, in the absence of forces, $\hat{H}$ must be of the form

$$\hat{H} = \alpha_1\hat{p}_1 c + \alpha_2\hat{p}_2 c + \alpha_3\hat{p}_3 c + \beta mc^2 \tag{42}$$

with $\hat{p}_i = -i\hbar\frac{\partial}{\partial x^i}$. The α's and β are dimensionless coefficients. They must be *dimensionless* because $p_i c$ and mc^2 have the dimension of $\hat{H}$. They must also be independent of $\hat{p}_i$ to ensure the linear dependence of $\hat{H}$ on the momenta, and they must not depend on the *coordinates* as this would introduce forces. This implies that they commute with the momentum operators.

[1]P.A.M. Dirac, Principles of Quantum Mechanics, 4th edition, Oxford University Press, 1958, chapter 11

Further properties of these coefficients can be found by requiring that the iteration of the operator yields the Klein-Gordon equation[2]:

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left[\alpha_1 \hat{p}_1 c + \alpha_2 \hat{p}_2 c + \alpha_3 \hat{p}_3 c + \beta mc^2\right]^2 \psi = \left\{(\hat{p}_1 c)^2 + (\hat{p}_2 c)^2 + (\hat{p}_3 c)^2 + \left(mc^2\right)^2\right\} \psi \quad (43)$$

In evaluating $[\ldots]^2$ we must allow for the possibility that the α's and β do not commute with each other. In order to get agreement with the Klein-Gordon equation, *i.e.* to impose the right-hand equality (43), we find that the α's and β must satisfy the following relations:

$$\begin{aligned} \alpha_i^2 &= 1, \qquad \beta^2 = 1 \\ \alpha_i \alpha_j + \alpha_j \alpha_i &= 0 \quad \text{for} \quad i \neq j \\ \alpha_i \beta + \beta \alpha_i &= 0, \quad i = 1,\, 2,\, 3 \end{aligned} \quad (44)$$

This means that the different α's anticommute with each other and with β. It must be possible therefore to represent them by matrices; they are called *Dirac matrices.*
To ensure that all matrix products of Eq. (44) are defined they must be square matrices.
The traces of all Dirac matrices must vanish. This can be seen, for instance, by rewriting the last of Eqs. (44) in the form of $\alpha_i = -\beta \alpha_i \beta$ (using $\beta^2 = 1$), then taking the trace, and hence, remembering that $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$, we get $\mathrm{Tr}\,\alpha_i = -\mathrm{Tr}\,\alpha_i$, *i.e.* $\mathrm{Tr}\,\alpha_i = 0$.
From the hermiticity of $\hat{H}$ it follows that the Dirac matrices are hermitian. Therefore their eigenvalues are real. From the first of Eqs. (44) it follows that their only eigenvalues are $+1$ and -1. Then, since the trace of a square matrix is the sum of its eigenvalues, and since the Dirac matrices are traceless, it follows that they must be of even dimension.
For the sake of economy we attempt to find the lowest order in which the Dirac matrices can be represented. The representation in terms of two-by-two matrices is ruled out because we know that there are only three linearly independent traceless two-by-two matrices, for instance the Pauli matrices. But a representation in terms of four-by-four matrices is possible. Indeed, the complete set of linearly independent four-by-four matrices consists of sixteen matrices, one of which can be chosen to be the unit matrix and the other fifteen matrices to be traceless.

For most purposes one does not need an explicit representation of the Dirac matrices. However, when an explicit representation is needed, it is mostly convenient to use what has become known as the *standard representation* of the Dirac matrices; partitioned into two-by-two matrices this is of the following form:

$$\alpha_i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix}, \qquad i = 1,\, 2,\, 3, \qquad \beta = \begin{pmatrix} \sigma_0 & \mathbf{0} \\ \mathbf{0} & -\sigma_0 \end{pmatrix}, \quad (45)$$

where σ_0 is the two-by-two unit matrix, $\mathbf{0}$ is the two-by-two null matrix and σ_i, $i = 1,\, 2,\, 3$, are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (46)$$

[2]this is because the Klein-Gordon equation is just the relativistic energy-momentum relation $E^2 = \vec{p}^{\,2} + (mc^2)^2$ together with the quantum mechanical replacement $E \to i\hbar\partial/\partial t$, $\vec{p} \to -i\hbar\nabla$

Exercise 1.
Verify that the Dirac matrices (45) are hermitian and satisfy the properties (44).
[Recall that $\sigma_i \sigma_j = \delta_{ij} + \mathrm{i}\varepsilon_{ijk}\sigma_k$ and $\sigma_i^\dagger = \sigma_i$]

Having represented the Hamiltonian by a four-by-four matrix we are forced to represent the wave function by a column matrix with four components,

$$\psi(\mathrm{x}) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

or in partitioned form

$$\psi(\mathrm{x}) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

where φ and χ are two-component spinors. Such a function is called a four-component spinor.

7.2 Continuity equation.
To deduce a continuity equation, we must write down a second wave equation for the hermitian conjugate wave function $\psi^\dagger$:

$$i\hbar \frac{\partial \psi^\dagger(x)}{\partial t} = -\left(H\psi(x)\right)^\dagger \tag{47}$$

Then, if we premultiply Eq. (41) by $\psi^\dagger(x)$, postmultiply Eq. (47) by $\psi(x)$ and add the resulting equations, we get

$$i\hbar \frac{\partial\, \psi^\dagger \psi}{\partial t} = \psi^\dagger H \psi - (H\psi)^\dagger \psi = -i\hbar\, \nabla\cdot\left(\psi^\dagger \vec{\alpha}\, \psi\right)$$

or, with $\rho = \psi^\dagger \psi$ and $\vec{\jmath} = \psi^\dagger \vec{\alpha}\, \psi$,

$$\frac{\partial \rho(x)}{\partial t} + \nabla\cdot\vec{\jmath} = 0$$

The density ρ is positive definite; it can therefore be interpreted as a probability density. Then $\vec{\jmath}$ is a probability current density. It can be shown that ρ and $\vec{\jmath}$ transform under Lorentz transformations like the time and space components of a 4-vector. They can therefore be combined into the 4-vector j^μ:

$$j^\mu = (\rho, \vec{\jmath})$$

Then, if we also define the four-dimensional derivative operator $\partial^\mu = (\partial_t, -\nabla)$, we can write the continuity equation in the manifestly invariant form

$$\partial^\mu j_\mu = 0 \tag{48}$$

7.3 Covariant form of the Dirac equation.
It is customary to introduce the matrices

$$\gamma^0 = \beta, \qquad \vec{\gamma} = \beta \vec{\alpha}$$

and to rewrite the Dirac equation in the covariant form

$$\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi(x) = 0$$

and we note that the mass term appears again, as in the Klein-Gordon equation, in the form of the inverse Compton wave length.

Setting from now on $\hbar = c = 1$, and using the "Feynman slash" notation,[3] $\not{p} \equiv \gamma^\mu p_\mu$, we can rewrite the Dirac equation in the following form:

$$(i\not{\partial} - m)\psi(x) = 0 \tag{49}$$

7.4 Properties of the γ matrices.
Without proof we list here for reference the following basic properties of the γ matrices:

$$\begin{aligned} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu} \\ \gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0 \\ \mathrm{Tr}\,\gamma^\mu &= 0 \end{aligned} \tag{50}$$

In the standard representation, the γ matrices are of the following form:

$$\gamma^0 = \begin{pmatrix} \sigma_0 & \mathbf{0} \\ \mathbf{0} & -\sigma_0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ -\sigma_i & \mathbf{0} \end{pmatrix}, \qquad i = 1,\, 2,\, 3, \tag{51}$$

7.5 Adjoint equation.
The adjoint equation is obtained by taking the hermitian conjugate of the Dirac equation:

$$[(i\not{\partial} - m)\psi(x)]^\dagger = \psi^\dagger(i\overleftarrow{\not{\partial}} - m)^\dagger = \psi^\dagger(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_\mu - m) = 0$$

and if we use the hermiticity equation for the γ matrices from above and set $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$, then we get

$$\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0 \tag{52}$$

where it is understood that the derivative operator $\overleftarrow{\not{\partial}}$ acts to the left.

From the Dirac equation and its adjoint equation we can immediately get the continuity equation. To do this we premultiply Eq. (49) by $\bar{\psi}(x)$, postmultiply the adjoint equation (52) by $\psi(x)$ and add the two resulting equations. This directly leads to Eq. (48) with $j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$. That j^μ is a 4-vector can be established either by appealing to the quotient theorem of tensor analysis or directly by carrying out a Lorentz transformation (see, e.g., P.A.M. Dirac, *Principles of Quantum Mechanics*, 4th edition, section 68).

7.6 Plane wave solutions.
We can find the plane wave solutions of the Dirac equation if we substitute

$$\psi(x) = u(p)e^{-ip\cdot x} \tag{53}$$

[3] $\not{p}$ is pronounced "p slash".

which yields

$$(\not{p} - m)u(p) = 0, \qquad \not{p} = \gamma^\mu p_\mu \tag{54}$$

Using the standard representation of the γ matrices, Eq. (51), we have

$$\not{p} = \begin{pmatrix} E & -\sigma\cdot\vec{p} \\ \sigma\cdot\vec{p} & -E \end{pmatrix}$$

where the energy E is understood to be multiplied into the unit 2×2 matrix.

The function $u(p)$ is a column matrix with four components. Let us write it in the following partitioned form:

$$u(p) = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

Substituting this into Eq. (54) we get the following coupled equations:

$$\begin{aligned} \vec{\sigma}\cdot\vec{p}\,u_A - (E+m)u_B &= 0 \\ (E-m)u_A - \vec{\sigma}\cdot\vec{p}\,u_B &= 0 \end{aligned}$$

from which, if we eliminate first u_B and then u_A, we get

$$(E^2 - \vec{p}^{\,2} - m^2)u_{A,B} = 0$$

and hence the eigenvalues $E = \pm\sqrt{\vec{p}^{\,2} + m^2}$. The occurrence of negative energy eigenvalues had to be expected from our discussion of the solutions of the Klein-Gordon equation. The corresponding states are again interpreted as antiparticle states of positive energy travelling backwards in time.

The eigenvectors can be found by standard methods of matrix algebra. For $E > 0$ we find

$$u_s(p) = N \begin{pmatrix} \varphi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\varphi_s \end{pmatrix}, \qquad s = 1,\, 2$$

where $\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and N is a normalization factor. Similarly we get the negative energy solutions:

$$u_{s+2}(p) = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\varphi_s \\ \varphi_s \end{pmatrix}, \qquad s = 1,\, 2$$

It can be shown that the four solutions of the Dirac equation are mutually orthogonal, *i.e.* that

$$u_r^\dagger(p)u_s(p) = 0 \qquad \text{if } r \neq s$$

The standard choice of normalization is to demand that there be $2E$ particles in the unit volume:

$$\int_{\text{unit vol.}} \psi^\dagger(x)\psi(x)\, dV = 2E$$

which, with appropriate choice of phase, yields the normalization factor $N = \sqrt{E+m}$.

7.7 Antiparticles.

As was mentioned in the previous section, the negative energy solutions are interpreted as antiparticle states. Thus, let $E < 0$ and make the substitution $E \to -E$, $\vec{p} \to -\vec{p}$, hence, if $u_{s+2}(p)$ is the negative energy solution, we have

$$(-\not{p} - m)u_{s+2}(-p) = 0$$

or, if we set $v_s(p) = u_{s+2}(-p)$, then

$$(\not{p} + m)v_s(p) = 0$$

7.8 Completeness relations.

Important are the following completeness relations, which we can verify for instance by using the standard representation:

$$\sum_{s=1,2} u_s(p)\bar{u}_s(p) = \not{p} + m, \qquad \sum_{s=1,2} v_s(p)\bar{v}_s(p) = \not{p} - m \tag{55}$$

7.9 Helicity.

As we have seen in our discussion of the plane wave solutions of the Dirac equation, there are two linearly independent solutions for $E > 0$ and two solutions for $E < 0$, *i.e.* we have a two-fold degeneracy of solutions. This degeneracy implies that there is an additional degree of freedom. This can be interpreted as being the electron spin. Formally the degeneracy means that there is another operator which commutes with the Hamiltonian and with the momentum operator. Such an operator is

$$\vec{\Sigma}\cdot\hat{p} = \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}, \qquad \text{where } \hat{p} = \vec{p}/p \tag{56}$$

The operator

$$\frac{1}{2}\vec{\sigma}\cdot\hat{p}$$

is the component of spin in the direction of $\vec{p}$. It is called the *helicity* operator, its eigenvalues $\lambda = \pm 1/2$ are the helicities of the electron.

7.10 Bilinear covariants.

We have seen above that the bilinear form $\bar{\psi}\gamma^\mu\psi$ is a 4-vector. One can construct other bilinear forms $\bar{\psi}\Gamma^i\psi$, $i = 1, 2, \ldots 16$, which are scalars (V), tensors (T), axial vectors (A) and pseudoscalars (P):

matrix Γ^i	$\mathbf{1}$	γ^μ	$\sigma^{\mu\nu}$	$\gamma^5\gamma^\mu$	γ^5
$\bar{\psi}\Gamma^i\psi$	S	V	T	A	P

Here $\sigma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The matrix γ^5 plays an important role especially in the theory of weak interactions. The main properties of γ^5 are the following:

$$\begin{aligned} \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 &= 0 \\ \gamma^{5\dagger} &= \gamma^5, \qquad (\gamma^5)^2 = 1 \end{aligned}$$

These properties can be checked by direct calculations.

In standard representation γ^5 is of the form

$$\gamma^5 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}$$

8. Electrodynamics of spin-1/2 particles.

8.1 Transition matrix.

To describe the interaction of electrons with electromagnetic field we must replace the operator p^μ in the Dirac equation by $p^\mu + eA^\mu$, thus

$$(\not{p} + e \not{A} - m)\psi(x) = 0$$

or

$$(\not{p} - m)\psi(x) = \gamma^0 V \psi(x)$$

with $V = -e\gamma^0 \not{A}$. In this definition of the interaction potential V, the sign and the factor γ^0 are chosen such as to be consistent with the corresponding expression in the nonrelativistic limit. Thus, we can immediately write down the transition amplitude in the lowest order of perturbation theory:

$$T_{fi} = -i \int \psi_f^\dagger V \psi_i \, d^4x = ie \int \bar{\psi}_f \not{A} \psi_i \, d^4x = -i \int j^\mu A_\mu \, d^4x$$

where $j^\mu = -e\bar{\psi}_f \gamma^\mu \psi_i$ is the transition current. Assuming plane waves in the initial and final states, we get

$$j^\mu = -e \left(\bar{u}_f \gamma^\mu u_i\right) e^{-iqx}$$

where $q = p_i - p_f$ and $u_{i,f} \equiv u(p_{i,f}, s_{i,f})$.

It is interesting to note the following decomposition of the transition current:

$$j^\mu = \frac{1}{2m} \bar{u}_f \left[(p_i + p_f)^\mu - i\sigma^{\mu\nu}(p_i - p_f)_\nu\right] u_i$$

(Gordon decomposition). In the first term in brackets we recognize the transition current of spinless particles (cf. Eq. (37)), which describes the interaction of the electromagnetic field with the charge of the electron. The second term corresponds to the interaction with the electron's magnetic moment.

Reasoning as previously in the case of electromagnetic scattering of spinless particles, we can write down the transition amplitude in lowest order of perturbation theory for collisions of two unlike spin-1/2 particles, say e^- and μ^-, *i.e.* we consider the reaction

$$e^- \mu^- \to e^- \mu^- \tag{57}$$

We denote the 4-momenta of the incident electron and muon by p_1 and p_2, respectively, their spins by s_1 and s_2, and those of the outgoing particles by p_3, p_4, s_3 and s_4. To simplify the notation we shall write the spinors as $u_i \equiv u(p_i, s_i)$, $i = 1, 2, 3, 4$. The electron and muon currents are then of the form

$$j^\mu(e) = -e\,(\bar{u}_3\gamma^\mu u_1)\,e^{-iqx}, \qquad j^\mu(\mu) = -e\,(\bar{u}_4\gamma^\mu u_2)\,e^{iqx} \tag{58}$$

where $q = p_1 - p_3 = p_4 - p_2$, and for the transition amplitude we get

$$T_{fi} = -i\int d^4x\, j^\mu(e)\left(-\frac{1}{q^2}\right) j_\mu(\mu)$$

The integral gives us a four-dimensional δ function, and we get

$$T_{fi} = -i(2\pi)^4\delta(p_1 + p_2 - p_3 - p_4)\mathcal{M}$$

where the transition matrix element is defined by[4]

$$-i\mathcal{M} = [ij^\mu(e)]\left[-\frac{ig_{\mu\nu}}{q^2}\right][ij^\nu(\mu)] \tag{59}$$

The three factors in square brackets refer, respectively, to the electron transition current, photon propagator and muon transition current. These are the elements of the *Feynman diagram* that provides a graphical representation of the collision process (see Fig. 1a).

The remaining calculation to find the differential cross section proceeds by the same steps as we have done in the case of scattering of spinless particles, except that we must in addition pay attention to the spins of the particles.

We begin by taking the mod-squared of the matrix element $\mathcal{M}$:

$$|\mathcal{M}|^2 = \frac{1}{q^4}\left(j^\mu(e)j_\mu(\mu)\right)^*\left(j^\nu(e)j_\nu(\mu)\right) = \frac{1}{q^4}\left[j^{\mu\,*}(e)j^\nu(e)\right]\left[j^*_\mu(\mu)j_\nu(\mu)\right]$$

Consider separately the two factors in square brackets. They carry two Lorentz labels, and therefore they are tensors. The first one of these refers to the electron. We call it the electron tensor and denote it by $\mathbf{L}^{\mu\nu}_{s_1s_3}$, where we have made the spin dependence explicit by labeling the tensor with the spins of the incident and the scattered electrons. The second factor is the muon tensor $\mathbf{M}^{\mu\nu}_{s_2s_4}$. Both factors have the same structure, so it is enough to calculate one of them, for instance the electron tensor. Written in detail, this is of the following form:

$$\mathbf{L}^{\mu\nu}_{s_1s_3} = e^2\,(\bar{u}_3\gamma^\mu u_1)^\dagger\,(\bar{u}_3\gamma^\nu u_1) = e^2\left(u_1^\dagger\gamma^{\mu\,\dagger}\bar{u}_3^\dagger\right)(\bar{u}_3\gamma^\nu u_1)$$

but $\bar{u}^\dagger = \left(u^\dagger\gamma^0\right)^\dagger = \gamma^0 u$, and if we also use the identity $\gamma^\mu = \gamma^0\gamma^{\mu\,\dagger}\gamma^0$, then we get

$$\mathbf{L}^{\mu\nu}_{s_1s_3} = e^2\sum_{ijkl}\left(\bar{u}_{1\,i}\gamma^\mu_{ij}u_{3\,j}\right)\left(\bar{u}_{3\,k}\gamma^\nu_{kl}u_{1\,l}\right)$$

where we have written the matrix labels i, j, k and l explicitly and indicated the summation from 1 to 4. Thus the expression under the sum is a product of ordinary numbers whose order

[4] from now on the symbols j^μ will be understood to represent $\bar{u}\gamma^\mu u$ *without* the exponential factors.

is arbitrary. We can, for instance, take the last factor, $u_{1\,l}$, to the front of the product. Then we recognize that the resulting expression is the following trace:

$$\mathbf{L}^{\mu\nu}_{s_1 s_3} = e^2 \mathrm{Tr}\, u_1 \bar{u}_1\, \gamma^\mu\, u_3 \bar{u}_3\, \gamma^\nu$$

Now we have a choice. We can continue the calculation assuming definite spin states of the particles. This is meaningful if we want to describe experiments in which the spins of the particles are measured. Alternatively we can consider the more usual experiment, where the incident beam and target are unpolarized and all scattered particles are counted independently of their spin states. In this case we have to apply the procedure called *spin summation*. To describe the unpolarized beam and target, assuming random spin orientations of the particles, we must *average* over the spin states of the incident electron and muon; to account for the counting of all particles in the final state irrespective of their spin states we must *sum* over the spins of the scattered particles. Thus we get the *spin averaged* electron tensor

$$\begin{aligned} \overline{\mathbf{L}}^{\mu\nu} &= \frac{1}{2}\sum_{s_1 s_3} \mathbf{L}^{\mu\nu}_{s_1 s_3} = \frac{e^2}{2}\mathrm{Tr}\left(\sum_{s_1} u_1 \bar{u}_1\right)\gamma^\mu \left(\sum_{s_3} u_3 \bar{u}_3\right)\gamma^\nu \\ &= \frac{e^2}{2}\mathrm{Tr}\,\left(\not{p}_1 + m\right)\,\gamma^\mu\,\left(\not{p}_3 + m\right)\,\gamma^\nu \end{aligned}$$

where in the last step we have used the completeness relation, Eq. (55), of the Dirac spinors. Opening the brackets we get

$$\overline{\mathbf{L}}^{\mu\nu} = \frac{e^2}{2}\mathrm{Tr}\,\left[\not{p}_1 \gamma^\mu\, \not{p}_3 \gamma^\nu + m\left(\gamma^\mu\, \not{p}_3 \gamma^\nu + \not{p}_1 \gamma^\mu \gamma^\nu\right) + m^2 \gamma^\mu \gamma^\nu\right] \tag{60}$$

8.2 Trace theorems.

To make further progress in evaluating the traces of products of γ matrices we need now the following *trace theorems*:

$$\mathrm{Tr}\,\gamma^\mu = 0 \tag{61}$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu} \tag{62}$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\,\ldots\,\gamma^\omega = 0 \qquad \text{for odd numbers of factors} \tag{63}$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho = 4\left(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho} + g^{\mu\rho}g^{\nu\lambda}\right) \tag{64}$$

Proof of trace theorem (61): use the γ^5 matrix defined in section 7.10, remembering that γ^5 anticommutes with γ^μ, $\mu = 0$, 1, 2, 3, and that $(\gamma^5)^2 = 1$, hence $\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}\,\gamma^\mu(\gamma^5)^2 = \mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^5$, where in the last step we have used the property of traces: $\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$, which is generally true if both products AB and BA are defined and are square matrices. If we now commute γ^μ with one of the γ^5 factors, we get $-\mathrm{Tr}\,\gamma^\mu(\gamma^5)^2 = -\mathrm{Tr}\,\gamma^\mu$, and hence $\mathrm{Tr}\,\gamma^\mu = -\mathrm{Tr}\,\gamma^\mu = 0$.

Proof of trace theorem (62): we write $\gamma^\mu\gamma^\nu = \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu\right)$, apply the theorem $\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$ to one of the terms in brackets, then use the commutation relation of the γ matrices and note that the metric tensor $g^{\mu\nu}$ is multiplied by the 4×4 unit matrix whose trace is equal to 4.

Proof of trace theorem (63): we postmultiply the product under the trace by $(\gamma^5)^2$, take the last one of the γ^5 factors to the front, then commute it past the product. In each commutation we pick up a factor of -1, in total an odd number of such factors, hence $\mathrm{Tr}\,\gamma^\mu\gamma^\nu\ldots\gamma^\omega = -\mathrm{Tr}\,\gamma^\mu\gamma^\nu\ldots\gamma^\omega = 0$.
The proof of (64) is more involved but proceeds on similar lines as the proof of (62). Similarly one can prove trace theorems for products of more than four γ factors.

8.3 Completion of the calculation of the electron tensor.
Applying the trace theorems we see that the terms linear in the electron mass m drops out on account of being multiplied by traces of products of three γ factors. The remaining terms give

$$\begin{aligned}\overline{\mathbf{L}}^{\mu\nu} &= 2e^2\left[p_{1\alpha}p_{3\beta}\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right) + m^2 g^{\mu\nu}\right]\\ &= 2e^2\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + \left(m^2 - p_1{\cdot}p_3\right)g^{\mu\nu}\right]\end{aligned}$$

The factor multiplying the metric tensor is $q^2/2$; indeed, we have

$$q^2 = (p_1 - p_3)^2 = 2m^2 - 2p_1{\cdot}p_3$$

and hence the statement. Thus we have the following final form of the electron tensor:

$$\overline{\mathbf{L}}^{\mu\nu} = 2e^2\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu + \frac{1}{2}q^2 g^{\mu\nu}\right] \tag{65}$$

The muon tensor has the same structure as the electron tensor. We can immediately write down the expression for the spin-averaged muon tensor:

$$\overline{\mathbf{M}}^{\mu\nu} = 2e^2\left(p_2^\mu p_4^\nu + p_2^\nu p_4^\mu + \frac{1}{2}q^2 g^{\mu\nu}\right) \tag{66}$$

The next step is to contract the two tensors. This is straight forward and sufficiently simple, but we can simplify this calculation further by making use of current conservation, $\partial^\mu j_\mu = 0$. Indeed, substituting the plane wave expressions for the currents, Eq. (58), we see that $\partial^\mu j_\mu = -iq^\mu j_\mu$. Thus current conservation is expressed by $q^\mu j_\mu = 0$, and this in turn implies

$$q_\mu\overline{\mathbf{L}}^{\mu\nu} = q_\nu\overline{\mathbf{L}}^{\mu\nu} = 0 \tag{67}$$

We apply this result to our calculation by replacing in the muon tensor p_4 by $p_2 + q$, hence

$$\overline{\mathbf{M}}^{\mu\nu} = 2e^2\left[2p_2^\mu p_2^\nu + p_2^\nu q^\mu + p_2^\mu q^\nu + \frac{1}{2}q^2 g^{\mu\nu}\right]$$

and then applying Eq. (67). Therefore the muon tensor can be replaced by the *effective* muon tensor

$$\overline{\mathbf{M}}^{\mu\nu}_{\text{eff}} = 2e^2\left[2p_2^\mu p_2^\nu + \frac{1}{2}q^2 g^{\mu\nu}\right] \tag{68}$$

8.4 Cross section of elastic $e\mu$ scattering in the CMS.
After contraction of the tensors we have the following expression for the spin averaged mod-squared scattering amplitude:

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left(p_1{\cdot}p_2 p_3{\cdot}p_4 + p_1{\cdot}p_4 p_3{\cdot}p_2 - m^2 p_2{\cdot}p_4 - M^2 p_1{\cdot}p_3 + m^2M^2\right)$$

and if we express the scalar products in terms of Mandelstam variables we get

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{t^2}\left[s^2 + u^2 - 2\left(m^2 + M^2\right)\left(s + u - 3m^2 - 3M^2\right)\right]$$

We shall be interested mainly in the ultrarelativistic limit, which formally corresponds to $m = M = 0$, hence

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{t^2}\left(s^2 + u^2\right) \tag{69}$$

and hence the differential cross section in the CMS:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\frac{s^2 + u^2}{t^2} \tag{70}$$

Expressing t and u in terms of the CMS scattering angle θ, *i.e.* $t = -s\,\sin^2(\theta/2)$, $u = -s\,\cos^2(\theta/2)$, we get

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} \tag{71}$$

Qualitatively the θ dependence is similar to the one we found for elastic $\pi^+ K^+$ scattering, Eq. (40). In particlular, we find again the singularity at $\theta = 0$, characteristic of photon exchange.

Experimentally the reaction $e\mu \to e\mu$ is of no interest except in the following sense. A method to measure the electromagnetic form factors of pions is to expose a target to a pion beam. The elastic scattering of pions by the atomic electrons of the target material has a characteristic kinematics that allows the reaction $\pi e \to \pi e$ to be separated from collisions of pions with nucleons. However pion beams are always contaminated with muons, and the relatively small mass difference between pion and muon causes a serious problem of background which must be understood for a correct interpretation of the data.

8.5 Electron-positron annihilation into muon pairs.

Closely related to the process $e^-\mu^- \to e^-\mu^-$ is the reaction $e^+e^- \to \mu^+\mu^-$, *i.e.* electron-positron pair annihilation into muon pairs, see Fig. 1b. In the latter reaction, the incoming positron is equivalent to an electron travelling backwards in time. We give it therefore a 4-momentum $(-p_2)$ (we call it $(-p_2)$ rather than $(-p_3)$ because it is an *incoming* particle). Similarly the outgoing positive muon gets a 4-momentum $(-p_3)$. In other words, the replacement $p_2 \to -p_3$, $p_3 \to -p_2$ takes us from elastic collision to pair annihilation. This procedure is called *crossing*. We can check that under crossing the Mandelstam variables s and t are exchanged, the third variable u remaining unchanged. We can therefore immediately apply this procedure to the mod-squared matrix element (69), thus

$$\overline{|\mathcal{M}(e^+e^- \to \mu^+\mu^-)|^2} = \frac{2e^4}{s^2}\left(t^2 + u^2\right) \tag{72}$$

and hence the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right) \tag{73}$$

where we have used the CMS kinematical relations $s = 4E^2$, $t = -2E^2(1 - \cos\theta)$, $u = -2E^2(1 + \cos\theta)$.

The characteristic feature of this result, which is worth remembering, is the symmetry of the angular distribution about $\theta = 90^o$. This was to be expected since, after spin-averaging, the reaction is completely symmetric in the CMS. This symmetry breaks down at sufficiently high energies where the contribution of the weak interaction becomes noticeable, because the weak interaction violates parity.

If we integrate (73) over the angles θ and ϕ from 0 to π and from 0 to 2π, respectively, we get the total cross section. To convert it to ordinary units we also multiply the result by $(\hbar c)^2 \approx 0.4\,\text{GeV}^2\,\text{mbarn}$, thus

$$\sigma(e^+e^- \to \mu^+\mu^-) = (\hbar c)^2 \frac{4\pi\alpha^2}{3s} \tag{74}$$

This cross section has been measured on electron-positron colliders over a large range of energies, in particular at the collider PETRA with CMS energies $\sqrt{s}$ from 10 to nearly 40 GeV. The experimental results in this energy range confirm the theoretical result quantitatively. Such a good agreement between theory and experiment is not trivial since we have used in our calculation only the lowest order of perturbation theory.

Finally, a last comment concerning the procedure, *i.e.* crossing, by which we have arrived at the result for the pair annihilation process. This is a powerful method. However, in the present simple and straight forward case we would have had no difficulty in deriving the result by repeating the entire calculation from the beginning. The reader is encouraged to do that as an exercise.

8.6 Electron-muon elastic scattering in the LAB frame.

Recall that the invariant expression of the differential cross section is given by

$$d\sigma = \frac{1}{F}\,\overline{|\mathcal{M}|^2}\,dQ$$

where F is the flux factor, $\overline{|\mathcal{M}|^2}$ is the spin-averaged mod-squared matrix element and dQ is the Lorentz invariant phase space factor. We shall tackle separately the three parts of the calculation, the matrix element, the phase space and the flux factor, in the LAB frame.

(i) Matrix element of electron-muon elastics scattering in the LAB frame.

We go back to the exact formulas for the lepton tensors, neglecting only the electron mass but keeping the muon mass, which will be essential in this calculation. Thus we have

$$\overline{|\mathcal{M}|^2} = \frac{1}{q^4}\overline{\mathbf{L}}^{\mu\nu}\overline{\mathbf{M}}_{\mu\nu}$$

with $\overline{\mathbf{L}}^{\mu\nu}$ given by Eq. (65) and $\overline{\mathbf{M}}^{\mu\nu}$ by Eq. (66). To emphasise that we are now working in the LAB frame, let us denote the 4-momenta of the incoming and outgoing electron by k and k', respectively, and similarly those of the muon by p and p'. Then carrying out the contraction and remembering that $q^2 = 2m^2 - 2k{\cdot}k' = 2M^2 - 2p{\cdot}p'$, we get

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left[k{\cdot}p\,k'{\cdot}p' + k{\cdot}p'\,k'{\cdot}p - m^2 p{\cdot}p' - M^2 k{\cdot}k' + 2m^2M^2\right]$$

or, if we set the electron mass m equal to nought, and hence $k^2 = k'^2 = 0$, and therefore $q^2 = (k-k')^2 = -2k{\cdot}k'$, we get

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left\{2k{\cdot}p\,k'{\cdot}p' + \frac{1}{2}q^2\left[M^2 - (k-k'){\cdot}p\right]\right\}$$

In the LAB frame the muon is at rest, hence $p = (M,0,0,0)$, and if we denote the energies of the incident and the outgoing electrons by E and E', respectively, we get

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}2M^2EE'\left[1 + \frac{q^2}{4EE'} - \frac{q^2}{2M^2}\,\frac{M(E-E')}{2EE'}\right]$$

We can get a more useful form of this formula if we express the energy of the scattered electron in terms of the scattering angle. To do this we consider 4-momentum conservation:

$$k + p = k' + p'$$

hence

$$p'^2 = (k - k' + p)^2 = q^2 + M^2 + 2(k-k'){\cdot}p$$

hence, with $p'^2 = M^2$, we get

$$q^2 = -2M(E-E') \tag{75}$$

Furthermore, setting $k = (E,0,0,E)$ and $k' = (E', k'_x, k'_y, k'_z)$, we have

$$q^2 = -2EE'(1-\cos\theta) = -4EE'\sin^2\frac{\theta}{2} \tag{76}$$

where we have put $k'_z = E'\cos\theta$. Using these relations we can cast the result for the matrix element in the following final form:

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}2M^2EE'\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

(ii) Two-body phase space in the LAB frame.
The Lorentz invariant two-body phase space factor is

$$dQ = \frac{1}{(4\pi)^2}\,\delta^{(4)}\left(p_3 + p_4 - p_1 - p_2\right)\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}$$

and we have to carry out four of the six integrations to get rid of the four-dimensional δ function. We begin by integrating over the 3-momentum $\vec{p}_4$; this removes the three-dimensional δ function $\delta^{(3)}\left(\vec{p}_3 + \vec{p}_4 - \vec{p}_1 - \vec{p}_2\right)$ and leaves us with

$$dQ = \frac{1}{(4\pi)^2}\,\delta\left(E_3 + E_4 - E_1 - E_2\right)\frac{p_3^2dp_3d\Omega}{E_3E_4}$$

where we have expressed the three-dimensional differential d^3p_3 in polar coordinates with $d\Omega = \sin\theta d\theta d\phi$, where θ and ϕ are the polar angle and azimuth, respectively.

So far the expression is valid in any frame. At this point we specify the LAB frame by setting

$$E_1 = E, \qquad E_3 = E', \qquad E_2 = M, \qquad \text{and} \qquad E_4 = \sqrt{\vec{p}'^2 + M^2}$$

hence

$$dQ_{\text{LAB}} = \frac{1}{(4\pi)^2}\delta\left(E' + E_4 - E - M\right)\,\frac{E'dE'd\Omega}{E_4}$$

We note that our integration over $\vec{p}_4$ has already enforced momentum conservation. Therefore we have

$$\vec{p}' = \vec{k} - \vec{k}', \qquad \text{hence} \qquad \vec{p}'^2 = E^2 + E'^2 - 2EE'\cos\theta$$

Now consider the argument of the δ function. Let us denote it by $f(E')$, *i.e.*

$$f(E') = E' + \sqrt{\vec{p}'^2 + M^2} - E - M$$

Its zero corresponds to energy conservation (in addition to the momentum conservation, which is already enforced), in other words, it corresponds to 4-momentum conservation which in the LAB frame is expressed by Eqs. (75) and (76). To evaluate the integral over E' we rewrite the δ function in the form of

$$\delta(f(E')) = \left|\frac{df(E'_0)}{dE'}\right|^{-1}\delta(E' - E'_0)$$

where E'_0 is given by $f(E'_0) = 0$. Differentiating $f(E')$ w.r.t. E' we get

$$\frac{df(E')}{dE'} = 1 + \frac{E' - E\cos\theta}{E_4} = \frac{E_4 + E' - E\cos\theta}{E_4}$$

or, applying energy conservation, $E_4 + E' = E + M$,

$$\frac{df(E')}{dE'} = \frac{M + E(1-\cos\theta)}{E_4} = \frac{Mk}{E'E_4}$$

where in the last step we have once more used Eqs. (75) and (76). Thus finally we have

$$\delta(f(E')) = \frac{E'E_4}{Mk}\,\delta(E' - E'_0)$$

Now we can do the integral over E' and get

$$dQ_{\text{LAB}} = \frac{1}{(4\pi)^2}\,\frac{E_0'^2}{ME}\,d\Omega = \frac{1}{(4\pi)^2}\,\frac{2E'^2}{s - M^2}\,d\Omega$$

where in the final expression we have dropped the subscript of E'_0, which is now redundant, and used $s = M^2 + 2ME$

Note that in its final form the expression remains valid also for the case of quasielastic collisions, such as electroproduction of nucleon resonances, $ep \to eN^*$, if the mass M in the denominator is understood to be the mass of N^*.

(iii) The flux factor in the LAB frame is obtained from the invariant expression

$$4\sqrt{(p_1{\cdot}p_2)^2 - (m_1 m_2)^2}$$

which on account of $m_1 = 0$, $p_1 = (E, 00, E)$ and $p_2 = (M, 0, 0, 0)$ simplifies to $4EM$.

(iv) Differential cross section in the LAB frame.
Putting our results for the matrix element, the phase space factor and the flux factor together, we get for the differential cross section of elastic electron-muon scattering the following expression:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm LAB} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{E'}{E} \left(\cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2}\right) \tag{77}$$

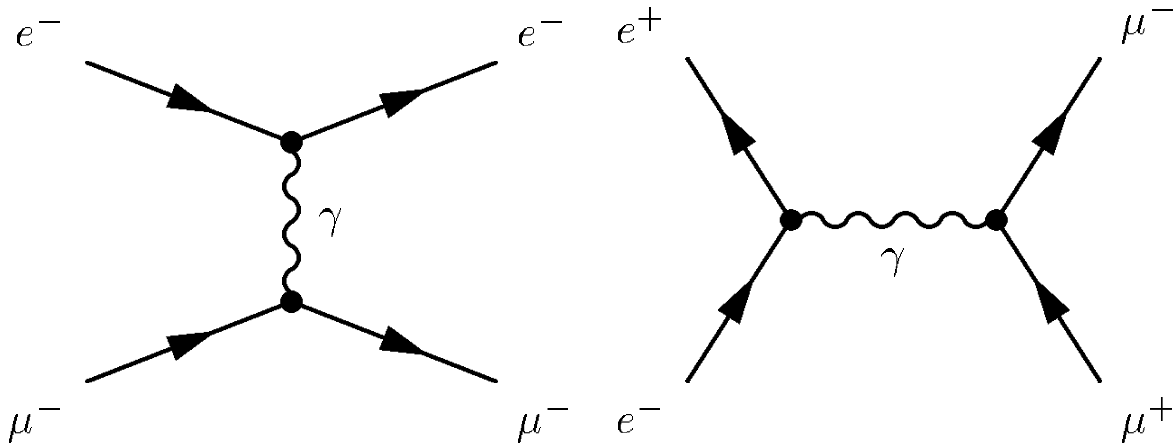

Figure 1: (a) Elastic $e^-\mu^-$ scattering; (b) $e^+e^- \to \mu^+\mu^-$ annihilation