The Two-Body Problem in Classical Mechanics

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Newton's equations of motion for two particles of masses m_1 and m_2, located at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, and interacting by gravitational attraction are, in the absence of external forces,

$$\frac{d\mathbf{p}_1}{dt} = -G\frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2), \qquad \frac{d\mathbf{p}_2}{dt} = +G\frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2), \tag{1}$$

where $\mathbf{p}_i = m_i\, d\mathbf{r}_i/dt$ is the momentum of particle i, $i = 1,\, 2$, and G is Newton's gravitational constant.

By adding the two equations, we get

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0 \tag{2}$$

or, with $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ denoting the the total momentum of the two-body system,

$$\mathbf{P} = \text{const.} \tag{3}$$

Thus we have found a first *conservation law*, namely the conservation of the total momentum of a two-body system in the absence of external forces. We can make use of this by carrying out a Galilei transformation to another inertial frame in which the total momentum is equal to zero. Indeed, let us apply the transformation

$$\mathbf{r}_i \to \mathbf{r}'_i = \mathbf{r}_i - \mathbf{v}t, \quad i = 1,\, 2 \tag{4}$$

hence $\mathbf{p}_i \to \mathbf{p}'_i = \mathbf{p}_i - m_i\mathbf{v}$ and hence, with $M = m_1 + m_2$,

$$\mathbf{P} \to \mathbf{P}' = \mathbf{P} - M\mathbf{v} \tag{5}$$

and if we choose $\mathbf{v} = \mathbf{P}/M$, then the total momentum is equal to zero in the primed frame. We also note that the gravitational force is invariant under the Galilei transformation, since it depends only on the difference $\mathbf{r}_1 - \mathbf{r}_2$. Thus let us from now on work in the primed frame, but drop the primes for convenience of notation. We can now replace the original equations of motion with the equivalent ones,

$$\mathbf{P} = 0, \qquad \frac{d\mathbf{p}}{dt} = -G\frac{m_1 m_2}{r^3}\mathbf{r} \tag{6}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $r = |\mathbf{r}|$, and $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$.

Next we introduce the position vector $\mathbf{R}$ of the centre of mass of the system:

$$\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2) \tag{7}$$

hence

$$\mathbf{P} = M\frac{d\mathbf{R}}{dt} \tag{8}$$

and hence from $\mathbf{P} = 0$ we have

$$\mathbf{R} = \text{const.} \tag{9}$$

and we can carry out a translation of the origin of our coordinate frame such that $\mathbf{R} = 0$. The coordinate frame we have arrived at is called *centre-of-mass frame* (CMS). We can also see now that

$$\mathbf{p} = \mathbf{p}_1 = m_1\frac{d\mathbf{r}_1}{dt} = m\frac{d\mathbf{r}}{dt}$$

where $m = m_1m_2/(m_1 + m_2)$ is the *reduced mass* of the system, and hence the equation of motion can be cast in the form of

$$m\frac{d^2\mathbf{r}}{dt^2} = -G\frac{mM}{r^2}\hat{\mathbf{r}} \tag{10}$$

where we have defined the radial unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$.

We can get two more conservation laws if we take the scalar product of Eq. (10) with $d\mathbf{r}/dt$ and its vector product with $\mathbf{r}$.

The scalar product with $d\mathbf{r}/dt$ gives on the left-hand side

$$\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2} = \frac{1}{2}\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2$$

and on the right-hand side we have

$$\frac{\hat{\mathbf{r}}}{r^2}\cdot\frac{d\mathbf{r}}{dt} = -\frac{d}{dt}\left(\frac{1}{r}\right)$$

hence

$$\frac{d}{dt}\left(\frac{\mathbf{p}^2}{2m} - \frac{\gamma}{r}\right) = 0 \tag{11}$$

where $\gamma = GmM$. This implies that the expression in brackets is conserved, *i.e.*

$$\frac{\mathbf{p}^2}{2m} - \frac{\gamma}{r} = E = \text{const.} \tag{12}$$

Here the first term is the kinetic energy, the second term is the potential energy, and the sum of kinetic energy and potential energy is the total energy E, which is a constant of motion. Now take the cross product of Eq. (10) with $\mathbf{r}$: on the right-hand side we get the cross product of collinear vectors, which is equal to zero, hence

$$\mathbf{r}\times m\frac{d^2\mathbf{r}}{dt^2} = m\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}\left(\mathbf{r}\times\mathbf{p}\right) = 0 \tag{13}$$

and hence, if we define the *angular momentum* $\mathbf{L}$ by

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} \tag{14}$$

we get the result

$$\frac{d\mathbf{L}}{dt} = 0, \qquad \text{or} \quad \mathbf{L} = \text{const.} \tag{15}$$

i.e. conservation of angular momentum. An immediate consequence of this conservation law is that the radius vector $\mathbf{r}$ always stays in one plane, namely the plane perpendicular to $\mathbf{L}$. This implies that we can without loss of generality choose this plane as the (xy) coordinate plane. The vector $\mathbf{r}$ is then a two-dimensional vector,

$$\mathbf{r} = (x,\, y) = r\hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = (\cos\phi,\, \sin\phi) \tag{16}$$

where we have defined the polar angle ϕ. With this notation we can express the magnitude of angular momentum as

$$L = mr^2 \frac{d\phi}{dt} \tag{17}$$

The conservation of angular momentum can be used to simplify the equation of motion (12). To do this we note that

$$\mathbf{L}^2 = (\mathbf{r} \times \mathbf{p})^2 = \mathbf{r}^2\mathbf{p}^2 - (\mathbf{r}\cdot\mathbf{p})^2$$

hence

$$\mathbf{p}^2 = \frac{\mathbf{L}^2}{r^2} + p_r^2$$

where $p_r = \hat{\mathbf{r}} \cdot \mathbf{p}$ is the radial component of momentum. Substituting into Eq. (12) then gives

$$\frac{p_r^2}{2m} + \frac{\mathbf{L}^2}{2mr^2} - \frac{\gamma}{r} = E$$

or, with $p_r = m\,dr/dt$,

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{\mathbf{L}^2}{2mr^2} - \frac{\gamma}{r} = E \tag{18}$$

Looking back at our starting point, Eq. (1), we see that we have achieved the following: we have reduced the simultaneous differential equations of six functions of time, namely the six components of the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, to a pair of simultaneous differential equations for the polar coordinates $r(t)$ and $\phi(t)$; these equations contain two constants of motion, the total energy E and angular momentum $\mathbf{L}$.

We have also split the kinetic energy into two terms of which, by virtue of the conservation of angular momentum, the second one is a function of r only. This term is therefore more similar to a potential energy. In fact, it is useful to consider it together with the potential energy and introduce the concept of an *effective* potential energy V_{eff},

$$V_{eff} = \frac{\mathbf{L}^2}{2mr^2} - \frac{\gamma}{r} \tag{19}$$

We note that the first term drops monotonically from $+\infty$ at $r = 0$ to zero for $r \to \infty$, *i.e.* it corresponds to a repulsive force, called the *angular momentum barrier*. The second term, the gravitational attraction, is less than zero everywhere, increasing monotonically from $-\infty$ at $r = 0$ to zero for $r \to \infty$. But since the gravitational attraction grows in magnitude more slowly with $r \to 0$ than the angular momentum barrier, except for $L = 0$, the net result is that the effective potential energy tends to $+\infty$ as $r \to 0$. The interplay between attraction and repulsion is such that the effective potential energy has a minimum. Indeed, differentiating with respect to r we find that the minimum lies at $r_0 = L^2/\gamma m$ and that

$$V_{eff}^{min} = -\frac{m\gamma^2}{2L^2}$$

Therefore, since the radial part of the kinetic energy, $T_r = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2$, is non-negative, the total energy must be not less than V_{eff}^{min}, *i.e.*

$$E \geq E_{min} = -\frac{m\gamma^2}{2L^2} \tag{20}$$

where the equal sign corresponds to $T_r = 0$, *i.e.* to no radial motion or, in other words, to circular motion. For $E_{min} < E < 0$ the trajectory lies between a smallest value r_{min} and a greatest value r_{max} which can be found from the condition $E = V_{eff}$, *i.e.*

$$r_{min,\,max} = -\frac{\gamma}{2E} \pm \sqrt{\left(\frac{\gamma}{2E}\right)^2 + \frac{L^2}{2mE}}$$

where the upper (lower) sign corresponds to r_{max} (r_{min}). For $E > 0$ only the upper sign gives an acceptable value; the second root is negative and must be rejected.

Let us now proceed solving the simultaneous differential equations (17) and (18). We can begin, for instance, by solving Eq. (18), which has no ϕ dependence, then substitute $r(t)$ into Eq. (17), which is solved by a single quadrature giving us $\phi(t)$. This gives us the equation of the trajectory in parametric form.

Alternatively we can eliminate the time from Eq. (18) by using Eq. (17) and solve the resulting differential equation, which gives us the equation of the trajectory in the form of $r = r(\phi)$. Taking the latter approach we have from Eq. (17)

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{L}{mr^2}\frac{dr}{d\phi} = -\frac{L}{m}\frac{d}{d\phi}\left(\frac{1}{r}\right) \tag{21}$$

and if we define the auxiliary variable $u = 1/r$, then Eq. (18) takes on the form

$$u'^2 + u^2 - \frac{2\gamma m}{L^2}u = \frac{2mE}{L^2} \tag{22}$$

where $u' = du/d\phi$ and we have divided by $L^2/2m$. Differentiating with respect to ϕ, we get

$$u'\left(u'' + u - \gamma m/L^2\right) = 0$$

hence either $u' = 0$, corresponding to the circular motion which we have discussed above, or

$$u'' + u = \gamma m/L^2 \tag{23}$$

which has the solution $u = \gamma m/L^2 + A\cos(\phi + \alpha)$ or, if we revert to the variable r,

$$r = \left[\frac{\gamma m}{L^2} + A\cos(\phi + \alpha)\right]^{-1} \tag{24}$$

which is the canonical form of conic sections in polar coordinates. The constants A and α have appeared as the two integration constants of the second oder differential equation (23). But this equation is an artifice of our derivation. The solution (24) must satisfy the first order differential equation (22). Substituting (24) into (22) we find, after a little algebra,

$$A^2 = \frac{2mE}{L^2} + \left(\frac{\gamma m}{L^2}\right)^2 \tag{25}$$

and therefore, taking account of Eq. (20), we get $A^2 \geq 0$. This implies that we have solutions of four classes:

$$1.)\ \text{A=0};\quad 2.)\ 0<|A|<\gamma m/L^2;\quad 3.)\ |A|=\gamma m/L^2;\quad 4.)\ |A|>\gamma m/L^2.$$

corresponding to $E = E_{min}$, $E_{min} < E < 0$, $E = 0$ and $E > 0$, respectively.

Solution 1: This is the solution corresponding to circular motion, $u' = 0$. We can use this condition to reproduce the result $r_0 = L^2/m\gamma$ that we have previously found for the value of r at which V_{eff} has its minimum, thus verifying the consistency of our calculation. We also note that the expression for r_0 together with Eq. (20) gives

$$r_0 = -\frac{\gamma}{2E_{min}} \tag{26}$$

Going back to the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, and remembering that we work in the centre-of-mass frame, we get $\mathbf{r}_1 = (m_2/M)\,\mathbf{r}$, $\mathbf{r}_2 = -(m_1/M)\,\mathbf{r}$, and hence

$$r_1/r_2 = m_2/m_1$$

Thus the two bodies move in concentric circles with radii, inversely proportional to their masses, and are always in opposition.

Solution 2: For $0 < |A| < \gamma m/L^2$, r remains finite for all values of ϕ. Since also $r(\phi+2\pi) = r(\phi)$, the trajectory is closed. In fact, it is an ellipse, as one knows from the theory of conic sections. If we choose $\alpha = 0$, then the major axis of the ellipse corresponds to $\phi = 0$. We get

$$r_{|\phi=0} = r_{min} = \left[\frac{\gamma m}{L^2} + A\right]^{-1} \qquad \text{and} \qquad r_{|\phi=\pi} = r_{max} = \left[\frac{\gamma m}{L^2} - A\right]^{-1} \tag{27}$$

and since $r_{max} + r_{min} = 2a$, where a is the *semi-major axis* of the ellipse, we get

$$a = \frac{\gamma m}{L^2}\left[\left(\frac{\gamma m}{L^2}\right)^2 - A^2\right]^{-1}$$

We can now eliminate A from the latter equation and Eq. (25) and hence get the following relation between a and the energy:

$$a = -\gamma/2E \tag{28}$$

Futhermore, if we denote the distance $r_{|\phi=\pi/2}$ by ℓ, called *semi-latus rectum* or *parameter* of the ellipse, then we get

$$\ell = L^2/\gamma m \tag{29}$$

and hence the equation of the trajectory in the form of

$$r = \frac{\ell}{1+\varepsilon\cos\phi} \tag{30}$$

where $\varepsilon = \sqrt{1-\ell/a}$ is the *eccentricity* of the ellipse.
This completes our discussion of the trajectory of planetary motion.

Let us now consider once more the conservation of angular momentum, Eq. (15) together with (17). If we divide Eq.(17) by $2m$ and multiply by dt, we get

$$dF = \frac{1}{2}r^2\,d\phi = \frac{L}{2m}dt \tag{31}$$

which is the area swept by the radius vector r in the time interval dt, and since by Eq. (15) $L/2m$ is constant, we can integrate to get the area swept by the radius vector during the finite interval of time t: $F(t) = (L/2m)t$, and if we set t equal to the time of one full revolution, T say, then F is the area of the ellipse, *i.e.* $F = \pi ab$, where b is the minor semi-axis. From the theory of conic sections we know that $b = \sqrt{a\,\ell}$. Using also the relation between L and ℓ found above, Eq. (29), and remembering that $\gamma = GmM$, we can then deduce the following equation:

$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2} \tag{32}$$

If we want to apply this result to the solar system, we can now invoke Kepler's 3rd law, by which the ratio a^3/T^2 is constant for all planets. But M is the total mass of the two-body system, *i.e.* in the case of the solar system $M = M_\odot + m_p$, where $M_\odot$ is the mass of the sun and m_p is the mass of a planet. We must therefore conclude that the masses of the planets are immeasurably small in comparison with the mass of the sun, so that M represents, within observational accuracy, the solar mass $M_\odot$. The values of a and T are obtained for all planets by astronomical observations; Newton's gravitational constant G is found in laboratory experiments with a torsion balance. Therefore Eq. (32) allows us to find $M_\odot$. Taking, for instance, the values appropriate for the Earth,[1] *i.e.* $a = 1AU = 1.4960 \times 10^{11}\,\mathrm{m}$ and $T = 1\,\mathrm{yr} = 3.1557 \times 10^7\,\mathrm{s}$, together with $G = 6.6726 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$, we get for the solar mass

$$M_\odot = 1.9893 \times 10^{30}\,\mathrm{kg}$$

Similarly we can get the masses of planets whose satellites have masses neglibly small compared with the mass of their planet. The criterion is again Kepler's third law, whose validity can be checked if the planet has several satellites.[2] The problem of determining the masses of planets which have no more than one satellite and of determining the masses of the satellites themselves is outside the scope of these notes.

Solutions 3 and 4: We discuss these solution together. They correspond to $E \geq 0$ or, equivalently, $|A| \geq \gamma m/L^2$.

Repeating the calculation for solution 2, we find again the equation of the trajectory in the form of

$$r = \ell\,(1 + \varepsilon \cos\phi)^{-1} \tag{33}$$

but now $\varepsilon \geq 1$, where the equal sign corresponds to $E = 0$. Therefore, in order to ensure positivity of r, the polar angle ϕ must now be restricted to the interval given by

$$1 + \varepsilon \cos\phi > 0$$

or

$$\cos\phi > -1/\varepsilon \tag{34}$$

For solution 3 this means $\cos\phi > -1$, *i.e.* $\phi \in (-\pi,\, \pi)$. The trajectory is therefore not closed any more. For $\phi \to \pm\pi$ we have $r \to \infty$. From the theory of conic sections it is known that the curve (33) with $\varepsilon = 1$ is a parabola. In the case of solution 4, *i.e.* $\varepsilon > 1$, the allowed interval of polar angles is smaller than $(-\pi,\, \pi)$, and the trajectory is a hyperbola. In astronomical terms such trajectories correspond to nonreturning comets.

[1] AU denotes the *astronomical unit*, *i.e.* the mean distance of the Earth from the Sun; T is the *tropical year*, defined as the time between successive passages of the Sun through the Earth's equatorial plane at the vernal equinox; these constants and the value of G are known with greater accuracy than quoted here; see, for instance, the *Review of Particle Physics*, The European Physical Journal **C 15** (2000) No. 1-4.

[2] Mars has two satellites, Jupiter has 16, Saturn 18 or possibly more, Uranus 17, Neptune 8.

Appendix:
Dates of the Major Contributors to the Development of the Science of Planetary Motion and their major works:

Claudius Ptolemæus 90(?) - 160(?), *Almagest*
Nicolaus Copernicus 1473 - 1543, *On the Revolutions of the Celestial Spheres* (1543)
Tycho Brahe 1546 - 1601
Galileo Galilei 1564 - 1642, *The Great Systems of the Universe*
Johannes Kepler 1571 - 1630, *The New Astronomy* (1609)
Isaac Newton 1642 - 1727, *Philosophiæ Naturalis Principia Mathematica*