Mathematical Techniques
Part 6. Fourier Expansion and Fourier Transfrom

Lecture notes by W.B. von Schlippe

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1. Fourier expansion of periodic functions.

1.1 Periodic functions

A function $f(x)$ is *periodic* if it has the property

$$f(x+L) = f(x) \tag{1}$$

for all x; the constant L is called the *period* of $f(x)$. Examples of periodic functions are the trigonometric functions $\cos x$ and $\sin x$ which have period 2π. A function with period L can be transformed into a function with period 2π by the substitution

$$x \to y = \frac{2\pi x}{L} \tag{2}$$

We shall therefore discuss the case of functions with period 2π, but always being aware that our discussion is generally valid for any periodic function.

Fundamental for the theory of Fourier expansions is the trigonometric polynomial

$$\Sigma_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \tag{3}$$

which is periodic with period 2π. In the next section we shall see that the trigonometric polynomial can be used to yield an approximate represention of an arbitrary periodic function of period 2π. As a preparation for the following derivations we establish here the important *orthogonality property* of the trigonometric functions: consider the integral

$$I_1 = \int_0^{2\pi} \cos mx \, \cos nx \, dx, \qquad \text{for } m \neq n \tag{4}$$

which is taken by applying the trigonometric identity

$$\cos\alpha \, \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

hence

$$I_1 = \frac{1}{2}\int_0^{2\pi} \left[\cos(m+n)x + \cos(m-n)x\right] \, dx = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi} = 0$$

On the other hand, for $m = n$ the integral is

$$I_1 = \frac{1}{2}\int_0^{2\pi} (1 + \cos 2mx) \, dx = \pi$$

We can summarise the result using the Kronecker δ symbol

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

hence

$$I_1 = \int_0^{2\pi} \cos mx \, \cos nx \, dx = \pi\delta_{mn} \tag{5}$$

Similarly we can establish the following results:

$$I_2 = \int_0^{2\pi} \sin mx \, \sin nx \, dx = \pi\delta_{mn} \tag{6}$$

$$I_3 = \int_0^{2\pi} \cos mx \, \sin nx \, dx = 0 \tag{7}$$

The property of trigonometric functions, expressed by Eqs. (5) to (7) for $m \neq n$, is called *orthogonality* and one says that the trigonometric functions of multiples of x are a set of mutually orthogonal functions.

1.2 Approximate representation of periodic functions by trigonometric polynomials; Fourier coefficients

Let $f(x)$ be a periodic function with period 2π, i.e.

$$f(x + 2\pi) = f(x) \tag{8}$$

and consider the following equation:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) + R_n(x) \tag{9}$$

Here we have on the r.h.s. the sum of the trigonometric polynomial

$$\Sigma_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \tag{10}$$

and a residual term $R_n(x)$. Our aim is to find expansion coefficients a_m and b_m such that $\Sigma(x)$ approximate $f(x)$ as closely as possible in the sense that the mean square deviation of $\Sigma(x)$ from $f(x)$ over one period be as small as possible:

$$\langle R_n^2 \rangle = \int_0^{2\pi} R_n^2(x) dx = \min \tag{11}$$

The necessary condition is given by

$$\frac{\partial \langle R_n^2 \rangle}{\partial a_m} = 0, \qquad \text{and} \qquad \frac{\partial \langle R_n^2 \rangle}{\partial b_m} = 0 \tag{12}$$

This condition is also sufficient because of the linear dependence of $R_n(x)$ on the expansion coefficients a_n, b_n, Thus if $m > 0$ we have

$$\begin{aligned} \frac{\partial \langle R_n^2 \rangle}{\partial a_m} &= \int_0^{2\pi} \frac{\partial R_n^2}{\partial a_m} dx = 2 \int_0^{2\pi} R_n \frac{\partial R_n}{\partial a_m} dx \\ &= 2 \int_0^{2\pi} [\Sigma(x) - f(x)] \frac{\partial \Sigma(x)}{\partial a_m} dx = 2 \int_0^{2\pi} [\Sigma(x) - f(x)] \cos\, mx dx \\ &= 2 \left[\int_0^{2\pi} \Sigma(x) \cos\, mx dx - \int_0^{2\pi} f(x) \cos\, mx dx \right] \\ &= 2 \left[\pi a_m - \int_0^{2\pi} f(x) \cos\, mx dx \right] = 0 \end{aligned} \tag{13}$$

hence

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx dx \tag{14}$$

and similarly we can show that Eq. (14) is valid also for a_0 (this was the reason for beginning the Fourier polynomial with $a_0/2$ rather than with a_0, which would have led to different formulas for a_0 and $a_m,\ m > 0$). Repeating the derivation also for b_m we find

$$b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx dx \tag{15}$$

It must be realised that in this derivation it was not important that we have taken the integrals over the interval $[0, 2\pi]$. Instead we could have chosen any interval of length 2π, in particular the interval $[-\pi, \pi]$. It is left as an exercise to repeat the derivation for this interval and hence to show that

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx \qquad \text{and} \qquad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx \tag{16}$$

The coefficients a_m and b_m, defined by Eqs. (14) and (15) (or by Eq. (16)) are called the *Fourier coefficients* of $f(x)$.

1.3 Example. Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [-\pi, 0] \\ x & \text{if } x \in [0, \pi] \end{cases} \qquad \text{periodic} \tag{17}$$

To find the Fourier coefficients we use Eq. (16), hence

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2} \\
a_m &= \frac{1}{\pi}\int_0^\pi x\,\cos mx\,dx = \frac{1}{m^2\pi}[\cos m\pi - 1] = -\frac{1}{m^2\pi}[1-(-1)^m] \\
b_m &= \frac{1}{\pi}\int_0^\pi x\,\sin mx\,dx = \frac{(-1)^m}{m}
\end{aligned}$$

and we have found the representation of $f(x)$ as

$$f(x) = \frac{\pi}{4} + \sum_{k=1}^{n}\left\{-\frac{1}{k^2\pi}[1-(-1)^k]\cos kx + \frac{(-1)^k}{k}\sin kx\right\} + R_n(x)$$

An interesting result follows if we set $x=0$: then $f(0)=0$, $\cos kx = 1$ and $\sin kx = 0$, hence

$$\frac{\pi^2}{4} = \sum_{k=1}^{n}\frac{1-(-1)^k}{k^2} + R_n(0)$$

or

$$\frac{\pi^2}{8} = \sum_{k=1}^{n}\frac{1}{(2k-1)^2} + R_n(0)$$

and if we let $n \to \infty$ we get a convergent series and $R_n(0) \to 0$, hence

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

By similar techniques one can find closed expressions for many other series: see section 3, Exercises and problems.

1.4 Even and odd functions; Fourier coefficients of even and odd functions

A function $f(x)$ is called an *even* or *symmetric* function if it has the property

$$f(-x) = f(x) \tag{18}$$

and is called an *odd* or *antisymmetric* function if

$$f(-x) = -f(x) \tag{19}$$

A function which is neither even nor odd can be represented as the sum of an even and an odd function. Indeed, let $f(x)$ be a function which is neither even nor odd, and let $g(x)$ be an even function and $h(x)$ an odd function, i.e. $g(-x) = g(x)$ and $h(-x) = -h(x)$, and let

$$f(x) = g(x) + h(x) \tag{20}$$

then, changing the sign of x, we have

$$f(-x) = g(-x) + h(-x) = g(x) - h(x) \tag{21}$$

then, if we add Eqs. (20) and (21), we get

$$g(x) = \frac{1}{2}(f(x) + f(-x))$$

and if we subtract Eq. (21) from (20) we get

$$h(x) = \frac{1}{2}(f(x) - f(-x))$$

and, moreover, we see that the even and odd parts are uniquely defined in terms of $f(x)$.

The property of even and odd functions which is of particular interest to us in connection with Fourier expansions, is their behaviour under integration over symmetric intervals. Thus, let $f_e(x)$ be an even function and consider its integral from $-a$ to a:

$$\int_{-a}^{a} f_e(x)dx = \int_{-a}^{0} f_e(x)dx + \int_{0}^{a} f_e(x)dx \tag{22}$$

and if we make the substitution $x \to -x$ in the integral from $-a$ to 0 and use the symmetry $f_e(-x) = f_e(x)$, we get

$$\int_{-a}^{a} f_e(x)dx = 2\int_{0}^{a} f_e(x)dx \tag{23}$$

Similarly we can show that for an odd function $f_o(x)$ the integral over a symmetric interval $[-a, a]$ vanishes:

$$\int_{-a}^{a} f_o(x)dx = 0 \tag{24}$$

Frequently we must take integrals over symmetric intervals of products of functions of definite symmetry. Then it is useful to be aware that the product of two even functions or of two odd functions is an even function, and the product of an even and an odd function is an odd function.

Important examples of functions with definite symmetry are the trigonometric functions. Thus, $\cos x$ is an even function and $\sin x$ is an odd function.

Consider now the Fourier expansion of the function $f(x)$. Its Fourier coefficients are given by Eqs. (16). Consider also the decomposition of $f(x)$ into its even and odd parts,

$$f(x) = f_e(x) + f_o(x), \qquad f_e(-x) = f_e(x), \qquad f_o(-x) = -f_o(x)$$

hence

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(f_e(x) + f_o(x))\cos mx dx = \frac{2}{\pi}\int_{0}^{\pi} f_e(x)\cos mx dx$$

and similarly

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx = \frac{2}{\pi}\int_{0}^{\pi} f_o(x)\sin mx dx$$

Thus the Fourier coefficients a_m are determined only by the even part of $f(x)$ and the b_m only by the odd part of $f(x)$. It follows, in particlular, that the Fourier expansion of an even function is a cosine series and the expansion of an odd function is a sine series:

$$\begin{aligned} f_e(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + R_n(x) \\ f_o(x) &= \sum_{k=1}^{n} b_k \sin kx + R_n(x) \end{aligned}$$

1.5 Fourier series; completeness of trigonometric functions

In practice one is always content with approximate representations of periodic functions by trigonometric polynomials. However from a mathematical point of view it is also of interest to consider the Fourier series which is obtained by letting $n \to \infty$ in the polynomial (9). One can show that for a wide class of periodic functions $f(x)$ the resulting series converges and the residue $R_n(x)$ tends to zero. These conditions are known as *Dirichlet's conditions.* They can be summarised as follows:

> *If the periodic function $f(x)$ has a finite number of minima, maxima and discontinuities in one period and if the integral of $|f(x)|$ over one period exists, then the Fourier series converges to $f(x)$ at every point of continuity; at a point a, at which $f(x)$ has a finite discontinuity, the Fourier series converges to* $\frac{1}{2}[f(a-\varepsilon)+f(a+\varepsilon)]_{|\varepsilon\to 0}$

Thus, for any periodic function $f(x)$ that satisfies Dirichlet's conditions we have

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{25}$$

with Fourier coefficients given by Eqs. (16).

The property of the set of trigonometric functions $\{\cos kx, \sin kx\}$, that any function $f(x)$ from a wide class of functions can be represented by a linear superposition of the elements of the set, is called its *completeness.* This property is by no means confined only to the trigonometric functions: especially in quantum mechanics one encounters many different examples of complete sets of functions.

1.6 Complex representation of Fourier series

When working with Fourier expansions of periodic functions it is usual to write the expansions in terms of trigonometric functions. Less common is the mathematically equivalent representation in terms of complex exponential functions. This alternative representation will be dicussed here as a preparation for our discussion of Fourier transforms, which are the subject of the next section.

Let us write down the Fourier expansion of the periodic function $f(x)$ of period 2π, then use Euler's formula expressing the trigonometric functions in terms of complex exponentials, and finally regroup the various terms:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \frac{e^{ikx}+e^{-ikx}}{2} + b_k \frac{e^{ikx}-e^{-ikx}}{2i} \right) \\ &= \frac{a_0}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \left[(a_k - ib_k) e^{ikx} + (a_k + ib_k) e^{-ikx} \right] \end{aligned} \tag{26}$$

then we set $c_k = (a_k - ib_k)/2$, hence

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} c_k e^{ikx} + \sum_{k=1}^{\infty} c_k^* e^{-ikx} \tag{27}$$

and make the substitution $k \to -k$ in the second sum on the r.h.s, hence

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} c_k e^{ikx} + \sum_{k=-\infty}^{-1} c_{-k}^* e^{ikx} = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{28}$$

where $c_0 = a_0/2$ and $c_{-k} = c_k^*$. Note that the latter relationship between the complex Fourier coefficients is a consequence of our tacit assumption of $f(x)$ being a real function. If we generalize the theory to complex functions $f(x)$, then there is no relationship between the coefficients c_{-k} and c_k.

To find the complex Fourier coefficients of a given periodic function $f(x)$ we can either first calculate the real Fourier coefficients a_k and b_k and use $c_k = c_{-k}^* = (a_k - ib_k)/2$, or we can directly proceed from Eq. (28): multiplying the equation by $\exp(-ik'x)$ and integrating over x from 0 to 2π we get

$$\int_0^{2\pi} f(x) e^{-ik'x} dx = \sum_{k=-\infty}^{\infty} c_k \int_0^{2\pi} e^{i(k-k')x} dx \tag{29}$$

On the r.h.s. all integrals vanish except the one with $k' = k$: indeed, setting $n = k - k'$ we get

$$\int_0^{2\pi} e^{inx} dx = \frac{1}{in} e^{inx}\Big|_0^{2\pi} = 0$$

if $n \neq 0$, but for $n = 0$ the integral reduces to

$$\int_0^{2\pi} dx = 2\pi$$

We can summarize our last two results in one formula:

$$\int_0^{2\pi} e^{i(k-k')x} dx = 2\pi \delta_{kk'} \tag{30}$$

where $\delta_{kk'} = 1$ if $k = k'$ and $= 0$ if $k \neq k'$ (Kronecker-δ). Substituting into Eq. (29) we get finally

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx \tag{31}$$

An interesting corollary of the complex representation of the Fourier expansion is obtained if we substitute the Fourier coefficient from Eq. (31) into Eq. (28), hence

$$f(x) = \sum_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} f(x') e^{-ikx'} dx' \right) e^{ikx}$$

and if we change the order of integration and summation, then

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x') \left(\sum_{-\infty}^{\infty} e^{ik(x-x')} \right) dx'$$

or, with the notation

$$\delta(x - x') = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{ik(x-x')} \tag{32}$$

we have the result

$$f(x) = \int_0^{2\pi} f(x') \delta(x - x')\, dx', \qquad x \in (0, 2\pi) \tag{33}$$

which is the definition of the Dirac δ function. Thus Eq. (32) is a representation of the δ function. It is closely related to the representation of the δ function in terms of a Fourier integral – see Section 2.2.

1.7 Fourier expansion as a mathematical representation of spectral analysis

Our discussion of the Fourier expansion of periodic functions has shown that any periodic function is a linear superposition of sinusoids of multiples of a basic frequency. The corresponding amplitudes are uniquely determined by the function itself. In experimental physics one has various procedures to decompose superpositions of sinusoidal waves into waves of the constituent frequencies. This is called spectral analysis or *spectroscopy*. For instance in optics one uses either a prism or a diffraction grating to decompose light into its spectral components. As a result the frequencies and the corresponding intensities are measured. If we represent the incoming light by the function $f(t)$ (now we use t as the argument to remind us that we are considering a wave which is periodic in time), then the Fourier representation gives us all frequencies contained in $f(t)$. The lowest frequency is called *fundamental* frequency, its multiples are called the *harmonics*. The amplitudes for each of the frequencies are given by the Fourier coefficients. The intensities are the squares of the amplitudes.

We conclude from this discussion that Fourier analysis is a mathematical model of spectral analysis.

2. Fourier transform

2.1 Spectral analysis in the case of continuous spectra; Fourier transform

The Fourier method can be extended also to nonperiodic functions. In physical applications this is of interest when there is a wave phenomenon with a continuous spectrum. In acoustics one makes the distinction between *sound*, which has a discrete spectrum, and *noise*, which has a continuous spectrum. In signal processing one is sometimes interested in a single pulse travelling down a communication cable. In quantum mechanics an electron is represented by a wave packet which can be understood as a superposition of sinusoidal wavelets with a continuous spectrum. All these situations can be analysed by a method similar to that studied in the preceeding section. The key to this analysis is the concept of the *Fourier transform*.

Consider a function $f(x)$. Its Fourier transform $g(k)$ is defined by

$$g(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx \tag{34}$$

There is nothing profound about this definition. If we multiply a function of x by a function of x and k and integrate over x, then we are bound to be left with a function of k. That's really all there is to Eq. (34). What makes this definition interesting is that we can get the inverse relationship of $f(x)$ in terms of $g(k)$. This is the contents of the *Fourier theorem* which we derive in the next section.

2.2 Fourier theorem

Consider the periodic function $f(x)$ with period L. We can transform this to a function of period 2π if we make the substitution $x \to 2\pi x/L$. In other words, we have for $f(x)$ a complex Fourier expansion

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx/L} \tag{35}$$

where

$$c_k = \frac{1}{2\pi}\int_{-L/2}^{L/2} f(x)e^{-2\pi ikx/L}dx \tag{36}$$

(cf. Eq. (31)).

Now denote $2\pi k/L$ by ω_k. The sum over k can be transformed into a sum over ω_k, but then the increment $\Delta k = 1$ (which is not written down by convention) becomes $\Delta\omega_k = 2\pi\Delta k/L$, hence

$$\begin{aligned} f(x) &= \sum_{\omega_k=-\infty}^{\infty} c(\omega_k)e^{i\omega_k x}\Delta\omega_k \\ c(\omega_k) &= \frac{1}{2\pi}\int_{-L/2}^{L/2} f(x)e^{-i\omega_k x}dx \end{aligned}$$

and finally we let $L \to \infty$ and $\Delta\omega_k \to d\omega$; the sum over ω_k then becomes an integral over ω. If we also change the notation writing $g(\omega)$ instead of $c(\omega)$ we get the Fourier theorem

$$f(x) = \int_{-\infty}^{\infty} g(\omega)e^{i\omega x}d\omega \tag{37}$$

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \tag{38}$$

In our derivation we have inherited the unsymmetric positioning of the $1/2\pi$ factor from the Fourier expansion. There is nothing to stop us from choosing a more symmetric notation, which is used by many authors, where the Fourier transform and its inverse both have a factor of $1/\sqrt{2\pi}$. At the same time we note that the symbol ω appears in Eqs. (37) and (38) as a dummy variable. It is more in keeping with the usual notation of physicists to use the symbol k as the partner of x – and think of it as a wave number – and to use ω as the partner of t, considering this pair as representing frequency[1] and time, respectively. In the latter case one speaks about the transformation between the *frequency domain* and the *time domain*. We can therefore write instead of Eqs. (37) and (38) the equivalent pair of equations

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}d\omega \tag{39}$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{40}$$

Finally we can also get the Fourier representation of the Dirac delta function if we substitute $f(t)$ from Eq. (39) into (40), hence

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(\omega')e^{i\omega' t}d\omega'\right)e^{-i\omega t}dt$$

and hence, changing the order of integration over t and ω', we get

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' g(\omega')\int_{-\infty}^{\infty} dt\, e^{i(\omega'-\omega)t} = \int_{-\infty}^{\infty} d\omega'\, g(\omega')\delta(\omega'-\omega)$$

where in the last step we have used the definition of the Dirac-δ function. Thus the Fourier representation of $\delta(\omega)$ is found to be

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t} \tag{41}$$

[1] strictly speaking *circular frequency*

2.3 Examples; matter waves and their representation by wave functions

Consider a wave pulse which, as a function of time, is represented by

$$f(t) = \begin{cases} h & \text{if } |t| < T \\ 0 & \text{otherwise} \end{cases}$$

Substituting into Eq. (40) we get

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} h e^{-i\omega t} dt = \frac{h}{i\sqrt{2\pi}\omega} \left(e^{i\omega T} - e^{-i\omega T} \right) = hT \sqrt{\frac{2}{\pi}} \frac{\sin \omega T}{\omega T} \tag{42}$$

Thus we have found that the Fourier transform of the "tophat" function $f(t)$ is, up to a constant, the *sinc* function, $\mathrm{sinc} z = (1/z) \sin z$.

Another interesting example is the Fourier transform of a Gaussian wave packet

$$f(x) = h e^{-x^2/2\sigma^2}$$

which is used as an example of a matter wave packet. The Fourier transform of $f(x)$ is

$$g(k) = \frac{h}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2 - ikx} dx$$

which is evaluated by rewriting the exponent in the form of

$$-\frac{k^2\sigma^2}{2} - \frac{1}{2} \left(\frac{x}{\sigma} + ik\sigma \right)^2$$

and making the substitution $x \to u = x/\sigma + ik\sigma$, hence

$$g(k) = h\sigma e^{-(k\sigma)^2/2}$$

The interesting result is that the Fourier transform of a Gaussian is again a Gaussian. There is also an interesting corollary to this result: we note that the parameter σ has the significance of the half-width of $f(x)$ at $1/\sqrt{e}$ of its maximum, and similarly $1/\sigma$ is the half-width of $g(k)$ at $1/\sqrt{e}$ of its maximum. These widths, denoted Δx and Δk, respectively, are also referred to as the *root-mean-square deviations* of their distributions. What is significant is that the product of Δx and Δk is independent of σ, more precisely

$$\Delta x \Delta k = 1.$$

In quantum mechanics a relation of this kind is known as *Heisenberg's uncertainty relation.* It must be realised that there is considerable latitude in the definition of the widths which we have used in this discussion. However, *any* consistent definition leads to the result that the product of the widths is independent of σ, and any *reasonable* definition of the widths results in a product which is a number of order 1 to 10, all of which are perfectly acceptable to quantum mechanics.

3. Exercises and problems

3.1) Find the Fourier expansion of the function

$$f(x) = \begin{cases} -1 & \text{if } x \in [-\pi, 0] \\ 1 & \text{if } x \in [0, \pi] \end{cases} \qquad \text{periodic}$$

hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$.

3.2) Find the Fourier expansion of the function

$$f(x) = |x| \text{ if } x \in [-\pi, \pi] \qquad \text{periodic}$$

hence deduce that $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{9} + \dfrac{1}{25} + \ldots + \dfrac{1}{(2n-1)^2} + \ldots$

3.3) Derive the formulas for the Fourier coefficients of a periodic function of period L.
Answer: $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{2\pi n x}{L}\,dx, \quad b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{2\pi n x}{L}\,dx$

3.4) Find the Fourier expansion of the function $f(x) = x^2$ for $x \in [-L/2, L/2]$, periodically continued with period L, and deduce a series representation of π^2.
Answer: The Fourier expansion of $f(x)$ is

$$f(x) = \frac{L^2}{12} + \frac{L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{2\pi n x}{L}$$

hence, putting $x = 0$, we get $\pi^2 = 12\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}/n^2$.

3.5) Find the Fourier expansion of the function $f(x) = 2x^2 - x^4$ for $x \in [-1, 1]$, periodically continued with period 2, and hence show that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

3.6) Find the Fourier transform of $f(x) = \left(x^2 + a^2\right)^{-1}$.
Note: reduce the Fourier integral to a manifestly real form, then consult a table of definite integrals.
Answer: $\displaystyle\int_{-\infty}^{\infty}\left(x^2 + a^2\right)^{-1}\exp(ikx)\,dx = (\pi/a)\exp(-ak)$.

3.7) Prove the following theorem: if $f(t)$ is a periodic function with period T and a_n and b_n are its Fourier coefficients, then

$$\int_0^T [f(t)]^2\,dt = \frac{T}{2}\left[\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)\right]$$