Mathematical Techniques:
Part 2. Polar, spherical and cylindrical coordinates

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The most widely used coordinates in physics applications are Cartesian coordinates. However other coordinate systems are frequently more convenient. Thus in two-dimensional problems with rotational symmetry it is frequently convenient to use polar coordinates, similarly in three-dimensional problems with spherical or cylindrical symmetry it is convenient to use spherical or cylindrical coordinates. Other coordinate systems are also sometimes used but will not be discussed here.

Polar coordinates

Consider a coordinate plane with Cartesian axes x (*abscissa*) and y (*ordinate*). Every point P in this plane is assigned a pair of numbers x and y: x is the distance of P from the y axis and y is its distance from the x axis. Denote the distance of P from the origin O by r. The line joining O and P makes an angle θ with the *positive* x axis. Thus the position of P in the coordinate plane can be characterized by specifying r and θ rather than x and y. In this sense r and θ are equivalent to x and y. They are called the polar coordinates of P. For the polar coordinate system the x axis becomes the *polar* axis. The y axis is not needed in the polar coordinate system.

If you drop a perpendicular from P onto the polar axis and denote the foot of the perpendicular by M, then the segment OM has the length x and the segment MP has the length y. From the triangle OMP we can read off the relation between the Cartesian and the polar coordinates of P:

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{1}$$

Solving these simultaneous equations for r and θ we find

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\frac{y}{x} \tag{2}$$

Assume that we are given a function in Cartesian coordinates:

$$y = f(x) \tag{3}$$

which represents a curve C in the (x, y) plane. In order to describe this curve in polar coordinates we can substitute $f(x)$ for y in Eq. (2); this gives the polar equation of the curve in parametric form:

$$r = \sqrt{x^2 + [f(x)]^2}, \qquad \theta = \arctan\frac{f(x)}{x} \tag{4}$$

If we eliminate x from this equation we get the polar equation of the curve in the form

$$r = r(\theta) \tag{5}$$

Of course we could have proceeded also by substituting x and y in terms of r and θ from Eq. (1) into Eq. (3) giving

$$r\sin\theta = f(r\cos\theta) \tag{6}$$

which is an equivalent form of the polar equation of curve C.

Similarly we can transform the polar equation of a curve to Cartesian coordinates.

Examples.
2.1) The equation of a circle of radius a about the origin is given in Cartesian coordinates by $\sqrt{x^2+y^2}=a$. The polar equation follows from substituting for x and y their expression in r and θ from Eq. (1), hence $r=a$. The polar angle θ does not appear in this equation: this means that the equation is true for *any* value of θ.
2.2) The polar equation of an ellipse is of the form $r=p(1-\varepsilon\cos\theta)^{-1}$ where $p=b^2/a$ is the *parameter* of the ellipse, $\varepsilon=f/a$ is the *eccentricity*, a and b are the *major* and *minor* semi-axes and $f=\sqrt{a^2-b^2}$ is the *focal distance*, i.e. the distance of the focus from the centre of the ellipse. In this form the origin of the polar coordinate system is at one of the foci of the ellipse and the polar axis lies along the major semi-axis. Assuming that the origin of the Cartesian coordinate system is at the centre of the ellipse and the x axis lies along the polar axis, the conversion from polar to Cartesian coordinates requires

$$x=f+r\cos\theta, \qquad y=r\sin\theta$$

and hence, after a few lines of calculations, one can get the *canonical* equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

2.3) Given a function $f(r,\theta)$ we can find its derivative w.r.t. x by applying the chain rule:

$$\frac{df(r,\theta)}{dx}=\frac{\partial f(r,\theta)}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial f(r,\theta)}{\partial \theta}\frac{\partial \theta}{\partial x}$$

and with

$$\frac{\partial r}{\partial x}=\frac{x}{r} \qquad \text{and} \qquad \frac{\partial \theta}{\partial x}=-\frac{y}{r^2}$$

we get

$$\frac{df(r,\theta)}{dx}=\frac{x}{r}\frac{\partial f(r,\theta)}{r}-\frac{y}{r^2}\frac{f(r,\theta)}{\partial\theta}$$

and similarly

$$\frac{df(r,\theta)}{dy}=\frac{y}{r}\frac{\partial f(r,\theta)}{r}+\frac{x}{r^2}\frac{f(r,\theta)}{\partial\theta}$$

2.4) The element of arc ds of the curve described by the function $y=f(x)$ is $ds=\sqrt{dx^2+dy^2}$. In polar coordinates it can be read off a sketch of the curve (see Fig. 2.1) which gives

$$ds=\sqrt{dr^2+r^2d\theta^2}.$$

This formula can be derived also by carrying out the transformation to polar coordinates of the formula for ds in Cartesian coordinates. This requires writing down the differentials dx and dy in terms of dr and $d\theta$:

$$dx=\frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial\theta}d\theta=\cos\theta dr-r\sin\theta d\theta, \qquad dy=\frac{\partial y}{\partial r}dr+\frac{\partial y}{\partial\theta}d\theta=\sin\theta dr+r\cos\theta d\theta$$

hence

$$ds^2=dx^2+dy^2=(\cos\theta dr-r\sin\theta d\theta)^2+(\sin\theta dr+r\cos\theta d\theta)^2=dr^2+r^2d\theta^2$$

2.5) The surface element dS in the polar plane can be read off Fig. 2.2:

$$dS = r dr d\theta$$

Spherical polar coordinates

Spherical polar coordinates are the three-dimensional generalization of polar coordinates appropriate for problems with spherical symmetry. Conventionally the z axis is chosen as the polar axis, the xy plane becomes the *equatorial* plane.

Consider a point P with Cartesian coordinates xyz. The line joining the origin O with P has the length r and makes an angle θ (*polar angle*) with the polar axis. Drop a perpendicular from P into the xy plane and denote its foot by M. The line OM in the xy plane makes an angle ϕ (*azimuth*) with the positive x axis. The spherical coordinates r, θ and ϕ are an alternative description of the position of P in space, equivalent with the Cartesian coordinates xyz. The ranges of r, θ and ϕ are

$$0 \le r < \infty, \qquad 0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi.$$

The relation between spherical and Cartesian coordinates can be read off Fig. 2.3. OQ is the projection of OP onto the z axis and we have

$$z = r\cos\theta$$

and $OM = r\sin\theta$; but ON is the projection of OM onto the x axis, hence

$$x = r\sin\theta\cos\phi \qquad \text{and} \qquad y = r\sin\theta\sin\phi.$$

Solving for r, θ and ϕ we get

$$r = \sqrt{x^2+y^2+z^2}, \qquad \theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}, \qquad \phi = \arctan\frac{y}{x}.$$

Frequently one needs the relations between the differentials dx, dy, dz and dr, $d\theta$, $d\phi$. These relations follow if we apply the general formula

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{\partial r}{\partial z}dz$$

with similar formulas for $d\theta$ and $d\phi$. Thus we need the nine partial derivatives $\partial r/\partial x$, $\partial r/\partial y$, ..., $\partial\phi/\partial z$:

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \sin\theta\cos\phi, & \frac{\partial r}{\partial y} &= \sin\theta\sin\phi, & \frac{\partial r}{\partial z} &= \cos\theta,\\
\frac{\partial \theta}{\partial x} &= \frac{1}{r}\cos\theta\cos\phi, & \frac{\partial \theta}{\partial y} &= \frac{1}{r}\cos\theta\sin\phi, & \frac{\partial \theta}{\partial z} &= -\frac{1}{r}\sin\theta,\\
\frac{\partial \phi}{\partial x} &= -\frac{\sin\phi}{r\sin\theta}, & \frac{\partial \phi}{\partial y} &= \frac{\cos\phi}{r\sin\theta}, & \frac{\partial \phi}{\partial z} &= 0.
\end{aligned}$$

Cylindrical coordinates

For problems with cylindrical symmetry one uses cylindrical coordinates z, ρ, ϕ, where z has the usual meaning of the z coordinate of a Cartesian coordinate system, and ρ and ϕ are polar coordinates in any plane perpendicular to the z axis. The transformation formulas relating Cartesian and cylindrical coordinates are

$$\rho = \sqrt{x^2 + y^2}, \quad \phi = \arctan \frac{y}{x}, \quad z = z$$

and the range of values of ρ and ϕ is

$$0 \leq \rho < \infty, \qquad 0 \leq \phi \leq 2\pi$$

The element of arc ds and volume element dV are in cylindrical coordinates given by

$$ds^2 = d\rho^2 + \rho^2 \, d\phi + dz^2, \qquad dV = \rho \, d\rho \, d\phi \, dz$$