Selected Topics in Physics

a lecture course for 1st year students

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Lecture 3

1. Particle in a Central Field
2. Kepler Problem

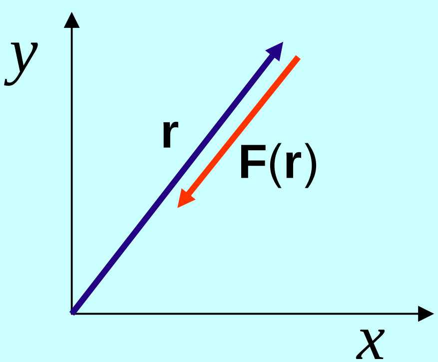
1. Particle in a Central Field

In 3D space, a force

$$\vec{F}(\vec{r}) = f(r)\hat{r}$$

where

$$r = |\vec{r}|, \quad \hat{r} = \vec{r}/r$$



is called a ***central force***.

A central force can be represented as the gradient of a function of the modulus of the vector **r**.

Consider the function $V(r)$

Its derivative w.r.t. x is

$$\frac{\partial V(r)}{\partial x} = \frac{dV(r)}{dr}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{dV(r)}{dr}$$

and similarly for the derivatives w.r.t y and z,

hence

$$\nabla V(r) \equiv \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right) = \frac{dV(r)}{dr} \frac{(x, y, z)}{r} = \frac{dV(r)}{dr} \hat{r} \qquad (3.1)$$

Thus we can represent the central force in the following form:

$$\vec{F}(\vec{r}) = -\nabla V(r)$$

where the minus sign is conventional.

Newton's equation of motion for a particle of mass m in a central force field is therefore of the following form:

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}(\vec{r}) = -\nabla V(r) \qquad (3.2)$$

$V(r)$ is the ***potential energy function***. The meaning of this terminology will become clear in the following derivation.

To derive **energy conservation**, take the scalar product ("***dot product***") with

$\dot{\vec{r}} = d\vec{r}/dt$ On the left-hand side we get

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} = \frac{d}{dt}\left(\frac{1}{2}\dot{\vec{r}}^{\,2}\right) \tag{3.3}$$

Indeed:

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{1}{2}\frac{d\dot{x}^2}{dt} + \frac{1}{2}\frac{d\dot{y}^2}{dt} + \frac{1}{2}\frac{d\dot{z}^2}{dt}$$

$$= \frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{d}{dt}\left(\frac{1}{2}\dot{\vec{r}}^{\,2}\right)$$

On the right-hand side we get

$$\dot{\vec{r}} \cdot \nabla V(r) = \frac{dV}{dt} \tag{3.4}$$

Indeed: from Eq. (3.1) we have

$$\dot{\vec{r}} \cdot \nabla V(r) = \frac{\dot{\vec{r}} \cdot \vec{r}}{r}\frac{dV}{dr}$$

also

$$\frac{dV(r)}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

$$= \frac{dV}{dr}\left(\dot{x}\frac{\partial r}{\partial x} + \dot{y}\frac{\partial r}{\partial y} + \dot{z}\frac{\partial r}{\partial z} \right) = \frac{\dot{\vec{r}} \cdot \vec{r}}{r}\frac{dV}{dr}$$

and comparing the r.h.s. with (3.5) we get the result (3.4).

Summarising our results we can write

$$\dot{\vec{r}} \cdot \left(m\ddot{\vec{r}} + \nabla V(r) \right) = \frac{d}{dt}\left(\frac{1}{2}m\dot{\vec{r}}^{\,2} + V(r) \right) = 0$$

hence

$$\boxed{\frac{1}{2}m\dot{\vec{r}}^{\,2} + V(r) = E = \text{constant.}} \qquad (3.5)$$

$T \equiv \frac{1}{2}m\dot{\vec{r}}^{\,2}$: Kinetic Energy $(K.E.)$ $\quad V(r)$: Potential Energy $(P.E.)$

We get a second conservation law if we take the vector product of (3.2) ("***cross product***") with vector r:

on the l.h.s. we get

$$\vec{r}\times\ddot{\vec{r}}=\frac{d}{dt}\left(\vec{r}\times\dot{\vec{r}}\right);$$

and on the r.h.s.:

$$\vec{r}\times\vec{F}=\vec{M}$$

this is called the ***moment of force*** or ***torque***. But since we are considering central forces, the torque is equal to zero:

$$\vec{r}\times f\left(r\right)\hat{r}=0$$

Now define the ***angular momentum***:

$$\boxed{\vec{L}\equiv m\vec{r}\times\dot{\vec{r}}}$$

Thus

$$\boxed{\frac{d\vec{L}}{dt}=0 \quad \text{and hence} \quad \vec{L}=\text{constant}} \tag{3.6}$$

Summary: Newton's equation of motion for a particle of mass m in a central force field

$$\vec{F}\left(\vec{r}\right)=f(r)\hat{r}$$

is of the form

$$m\frac{d^2\vec{r}}{dt^2}=\vec{F}\left(\vec{r}\right)=-\nabla V\left(r\right) \qquad (3.2)$$

where $V(r)$ is the *P.E.* fn, and we could derive two conservation laws:

(i) **conservation of energy:**

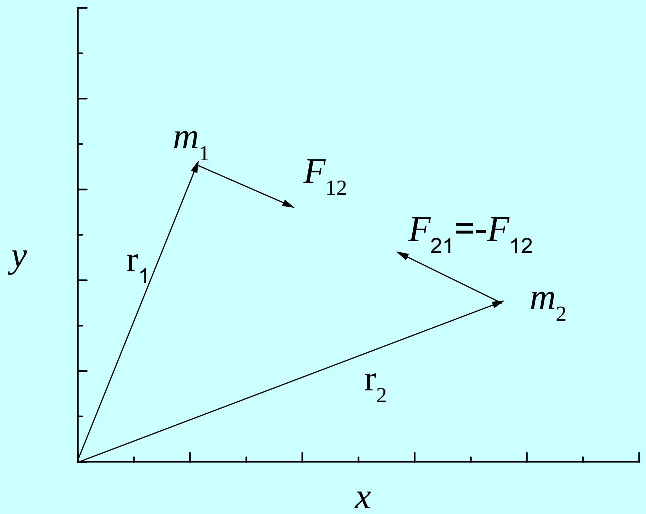
$$\frac{1}{2}m\dot{\vec{r}}^2+V\left(r\right)=E=\text{constant.} \qquad (3.5)$$

and (ii) **conservation of angular momentum:**

$$\vec{L}\equiv m\vec{r}\times\dot{\vec{r}}=\text{constant} \qquad (3.6)$$

2. Kepler Problem

The Kepler problem is the problem of the motion of a planet in the gravitational field of the sun or, more generally, the motion of two masses in gravitational interaction.



Thus consider two point masses, m_1 and m_2, located at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively.

The force exerted by m_2 on m_1 is

$$\vec{F}_{12} = G\frac{m_1 m_2}{\left|\vec{r}_1 - \vec{r}_2\right|^2}\frac{\vec{r}_2 - \vec{r}_1}{\left|\vec{r}_1 - \vec{r}_2\right|}$$

and the force exerted by m_1 on m_2 is by Newton's third law

$$\vec{F}_{21} = -\vec{F}_{12}$$

Thus we have the following two equations of motion for the two masses:

$$m_1\ddot{\vec{r}}_1 = G\frac{m_1 m_2}{\left|\vec{r}_1-\vec{r}_2\right|^3}\left(\vec{r}_2-\vec{r}_1\right) \qquad (3.7)$$

$$m_2\ddot{\vec{r}}_2 = -G\frac{m_1 m_2}{\left|\vec{r}_1-\vec{r}_2\right|^3}\left(\vec{r}_2-\vec{r}_1\right) \qquad (3.8)$$

If we add Eqs. (3.7) and (3.8), we get

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0 \qquad (3.9)$$

or with

$$\vec{p}_1 = m_1\dot{\vec{r}}_1; \quad \vec{p}_2 = m_2\dot{\vec{r}}_2$$

$$\frac{d}{dt}\left(\vec{p}_1+\vec{p}_2\right) = 0 \qquad (3.9a)$$

$\vec{P} = \vec{p}_1 + \vec{p}_2$ is the total momentum of the two-body system therefore our result (3.9) can be expressed by

$\vec{P} = \text{constant}$	***conservation of total momentum***

The radius vector of the centre of mass of m_1 and m_2 is defined by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad (3.10)$$

hence taking its derivative we find with Eq. (3.9):

$$\dot{\vec{R}} = \vec{P}/\left(m_1 + m_2\right) = \text{constant} \qquad (3.11)$$

and hence

$$M\ddot{\vec{R}} = 0 \qquad (3.12)$$

This equation is identical with Eq. (3.9) (or (3.9a)), but now we have a new interpretation: the entire two-particle system of mass M=m_1+m_2 is moving without acceleration.

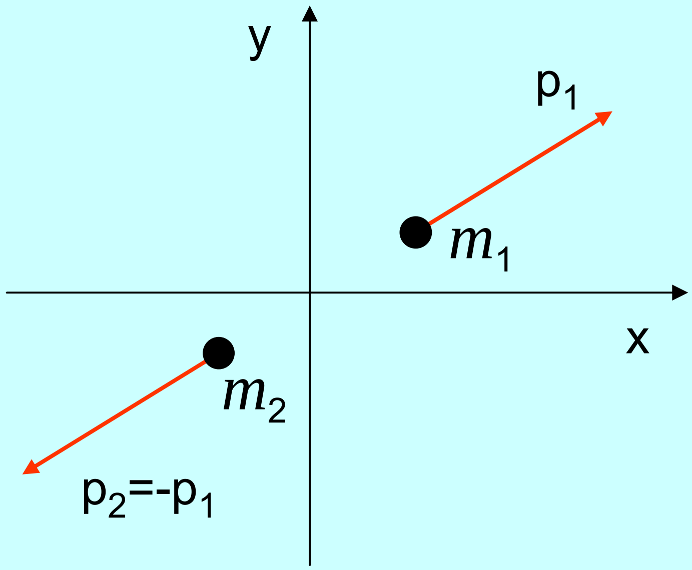
If we integrate Eq. (3.11), then we get

$$\vec{R} = \vec{R}_0 + \vec{v}t; \quad \vec{v} = \text{constant}$$

where $\vec{R}_0$ is an integration constant and $\vec{v} = \dot{\vec{R}}$

Carry out a GT into a frame in which the c.o.m. is at rest and at the origin (***Exercise***!): this is called the **centre of mass frame** or **centre of mass system (CMS)**.

This is illustrated on the next slide.



the centre of mass frame of particles m_1 and m_2

it is defined by

$$\vec{p}_2 = -\vec{p}_1$$

but the distance from the origin at any instant t is different:

$$\vec{r}_2(t) = -\frac{m_1}{m_2}\vec{r}_1(t)$$

in the CMS, Eq. (3.11) takes on the following form:

$$M\dot{\vec{R}} = 0$$

Let us go back to the equations of motion of the system of two interacting particles, but write it down for an arbitrary force. Taking account of Newton's third law we have:

$$m_1\ddot{\vec{r}}_1 = \vec{F} \qquad (3.13)$$

$$m_2\ddot{\vec{r}}_2 = -\vec{F} \qquad (3.14)$$

Divide (3.13) by m_1 and (3.14) by m_2, then subtract, hence

$$\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F} \qquad (3.15)$$

Now define

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$ *relative coordinate*

and

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$ *reduced mass μ*

then (3.15) becomes

$$\mu\ddot{\vec{r}} = \vec{F} \qquad (3.16)$$

and if we add (3.13) and (3.14), then as before we get Eq. (3.12):

$$M\ddot{\vec{R}} = 0 \qquad (3.12)$$

Thus our original equations (3.13) and (3.14) have been transformed into two simpler equations (3.12) and (3.16).
Equations (3.13) and (3.14) both depended on two vectors, $\mathbf{r}_1$ and $\mathbf{r}_2$.
They are therefore ***coupled equations***; in (3.12) and (3.16) we have *decoupled* them.

In the particular case, when the force **F** is the gravitational interaction,

$$\vec{F}_{12} = G\frac{m_1 m_2}{\left|\vec{r}_1 - \vec{r}_2\right|^2}\frac{\vec{r}_2 - \vec{r}_1}{\left|\vec{r}_1 - \vec{r}_2\right|}$$

we get

$$\mu\ddot{\vec{r}} = -G\frac{m_1 m_2}{r^2}\hat{r}$$

and if we note from the definition of the reduced mass that

$$m_1 m_2 = \mu\left(m_1 + m_2\right) = \mu M$$

then we get the equation of motion in the following form:

$$\boxed{\mu\ddot{\vec{r}} = -G\frac{\mu M}{r^2}\hat{r}} \qquad (3.17)$$

which is the equation of motion of a particle of mass μ in the gravitational field of a mass M that is at rest at the origin of the coordinate system.

Our task is now to solve Eq. (3.17), and then go back to the coordinates of the particles m_1 and m_2

Recall our results found for central forces:

- Conservation of energy, and
- conservation of angular momentum

$$\frac{1}{2}\mu\dot{\vec{r}}^{2}+V\left(r\right)=E=\text{constant.} \tag{3.18}$$

$$\frac{d\vec{L}}{dt}=0 \quad \text{and hence} \quad \vec{L}=\text{constant} \tag{3.19}$$

where $V(r)$ is defined by

$$\nabla V\left(r\right)=-\vec{F}\left(\vec{r}\right)=G\frac{\mu M}{r^{2}}\hat{r}$$

hence

$$V(r) = -\frac{\gamma}{r} \quad \text{with} \quad \gamma = G\mu M$$

and we remember that

$$\vec{L} = \mu \vec{r} \times \dot{\vec{r}}$$

Now we use conservation of angular momentum, Eq. (3.19), to simplify the expression (3.18) for the energy.

First we note that the conservation of angular momentum implies that the angular momentum vector has a ***constant direction***.

But the cross product of two vectors is perpendicular to the plane spanned by these vectors.

Therefore the vectors $\vec{r}$ and $\dot{\vec{r}}$ stay in one plane. Without …

… loss of generality we can make this the xy plane.
Therefore

$$\vec{r} = r\left(\cos\varphi, \sin\varphi, 0\right)$$

and its derivative is

$$\dot{\vec{r}} = \dot{r}\left(\cos\varphi, \sin\varphi, 0\right) + r\dot{\varphi}\left(-\sin\varphi, \cos\varphi, 0\right)$$

thus

$$\dot{\vec{r}}^{\,2} = \dot{r}^2 + r^2\dot{\varphi}^2 \quad \text{and} \quad \left|\vec{r} \times \dot{\vec{r}}\right| = r^2\dot{\varphi} \tag{3.20}$$

Recall:

$$T = \frac{1}{2}\mu\dot{\vec{r}}^{\,2} : \quad \text{Kinetic Energy} \quad (K.E.)$$

$$L^2 = \mu^2\left(\vec{r} \times \dot{\vec{r}}\right)^2 = \mu^2 r^4\dot{\varphi}^2 \tag{3.21}$$

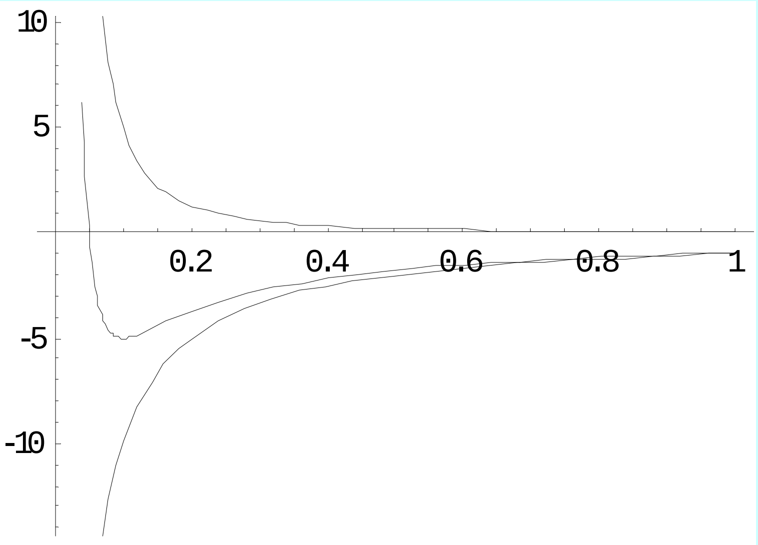
hence with (3.20):

$$T = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2}$$

hence

$$\frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{\gamma}{r} = E \qquad (3.22)$$

We have separated the K.E. into two terms. The first term depends on the radial velocity and the second term depends only on r. Thus the second term is, mathematically, similar to a potential energy. This leads us to the notion of an ***effective P.E***.:



$$V_{eff}(r) = \frac{L^2}{2\mu r^2} - \frac{\gamma}{r}$$

This fn has a minimum at

$$r_0 = L^2/\mu\gamma \qquad (3.23)$$

$$V_{eff}(r_0) = -\frac{\mu\gamma^2}{2L^2} \qquad (3.24)$$

The total energy cannot be less than the minimum of the effective P.E. since otherwise we would get a negative square of the radial velocity. Thus

$$E \geq -\frac{\mu\gamma^2}{2L^2}$$

The equal sign corresponds to zero radial velocity, *i.e*. to circular motion. The radius of the circular orbit is by Eq. (3.23) related to the energy E:

$$\boxed{r_0 = -\gamma/2E}$$

This is a remarkable result that we will remember when we come to discuss the Bohr theory of the hydrogen atom!

For $E_{\text{min}}<E<0$ the motion lies between a minimum and a maximum value of r, which we get from the condition

$$E = V_{eff}\left(r\right) \qquad \textit{i.e.} \qquad E = \frac{L^2}{2\mu r^2} - \frac{\gamma}{r}$$

Solving for r we get …

$$r_{\max,\min} = -\frac{\gamma}{2E} \pm \sqrt{\left(\frac{\gamma}{2E}\right)^2 + \frac{L^2}{2\mu E}}$$

where the upper (lower) sign corresponds to $r_{\max}$ ($r_{\min}$).

For E >0 only the upper sign gives an acceptable value; the second root must be rejected.

In astronomical terms E > 0 corresponds to the motion of a non-recurring comet.

Proceed now to solving Eq. (3.22):

$$\boxed{\frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{\gamma}{r} = E} \qquad (3.22)$$

To do this, recall Eq. (3.21) which we rewrite in the form of

$$L = \mu r^2 \dot{\varphi}$$

hence

$$\dot{\varphi} = \frac{L}{\mu r^2}$$

and then we eliminate the time by noting:

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\frac{L}{\mu r^2}$$

and substituting into (3.22) we get

$$\frac{L^2}{2\mu}\left(\frac{1}{r^2}\frac{dr}{d\varphi}\right)^2 + \frac{L^2}{2\mu r^2} - \frac{\gamma}{r} = E$$

or

$$\left(\frac{1}{r^2}\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{2\mu\gamma}{L^2}\frac{1}{r} = \frac{2\mu E}{L^2}$$

Now define $u = \frac{1}{r}$, hence $\frac{du}{d\varphi} = -\frac{1}{r^2}\frac{dr}{d\varphi}$

hence

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 - \frac{2\mu\gamma}{L^2}u = \frac{2\mu E}{L^2}$$

Differentiate w.r.t. φ

$$2u'u'' + 2uu' - \frac{2\mu\gamma}{L^2}u' = 0$$

Here u' is a common factor; therefore we have

either $(i)\; u' = 0,\quad \text{or}\; (ii)\; u'' + u = \frac{\mu\gamma}{L^2}$

Option (i) is the circular motion; consider option (ii): this DEq has the solution

$$u = \frac{1}{r} = \frac{\mu\gamma}{L^2} + A\cos(\varphi + \alpha)$$

and finally the equation of the trajectory:

$$r = \frac{1}{\frac{\mu\gamma}{L^2} + A\cos(\varphi + \alpha)} = \frac{L^2/\mu\gamma}{1 + A\left(L^2/\mu\gamma\right)\cos(\varphi + \alpha)}$$