Selected Topics in Physics

a lecture course for 1st year students

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Spring Semester 2007

Lecture 6

Relativity (continued)

Invariance of lengths of intervals.

Consider two events: **A** and **B**

Their coordinates are

$$\left(x_A, y_A, z_A, t_A\right) \quad \text{and} \left(x_B, y_B, z_B, t_B\right)$$

respectively.

We define the interval that separates the events:

$$\left(x_A - x_B, y_A - y_B, z_A - z_B, t_A - t_B\right)$$

then one can show that the ***Lorentz invariant square of the interval*** is given by

$$\left(\Delta s\right)^2 = \left(\Delta x\right)^2 + \left(\Delta y\right)^2 + \left(\Delta z\right)^2 - \left(c\Delta t\right)^2$$

where

$$\Delta x = x_A - x_B,\ \Delta y = y_A - y_B,\ \Delta z = z_A - z_B,\ \Delta t = t_A - t_B$$

Proof: assume for simplicity that we have

$$x \to x' = \gamma\left(x - vt\right),$$

$$y \to y' = y,$$

$$z \to z' = z$$

$$ct \to ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

hence

$$\left(\Delta s'\right)^2 = \left(\Delta x'\right)^2 + \left(\Delta y'\right)^2 + \left(\Delta z'\right)^2 - \left(c\Delta t'\right)^2$$

$$= \gamma^2\left(\Delta x - V\Delta t\right)^2 + \left(\Delta y\right)^2 + \left(\Delta z\right)^2 - \gamma^2\left(c\Delta t - \frac{V}{c}\Delta x\right)^2$$

$$= \gamma^2\left\{\left(1-\frac{V^2}{c^2}\right)\left(\Delta x\right)^2 - \left(1-\frac{V^2}{c^2}\right)\left(c\Delta t\right)^2 - 2\left[\left(V\right)-\left(V\right)\right]\Delta t\Delta x\right\}$$

$$+\left(\Delta y\right)^2 + \left(\Delta z\right)^2$$

but

$$\gamma^2 = 1\Big/\left(1-\frac{V^2}{c^2}\right)$$

hence

$$\left(\Delta s'\right)^2 = \left(\Delta x\right)^2 + \left(\Delta y\right)^2 + \left(\Delta z\right)^2 - \left(c\Delta t\right)^2 = \left(\Delta s\right)^2$$

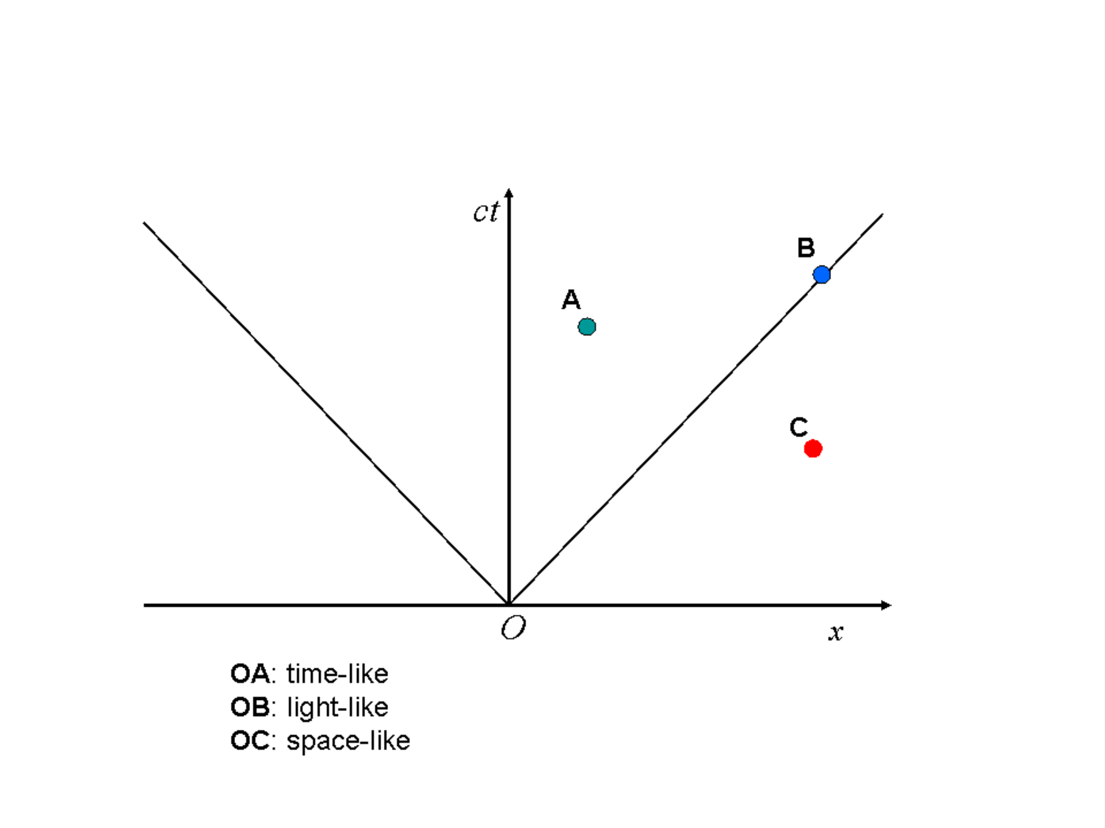
i.e. the square of the interval is Lorentz invariant.

The square of the interval between two events can be positive, negative or zero:

$$(\Delta s)^2 > 0: \quad \text{"space-like separation of events"}$$

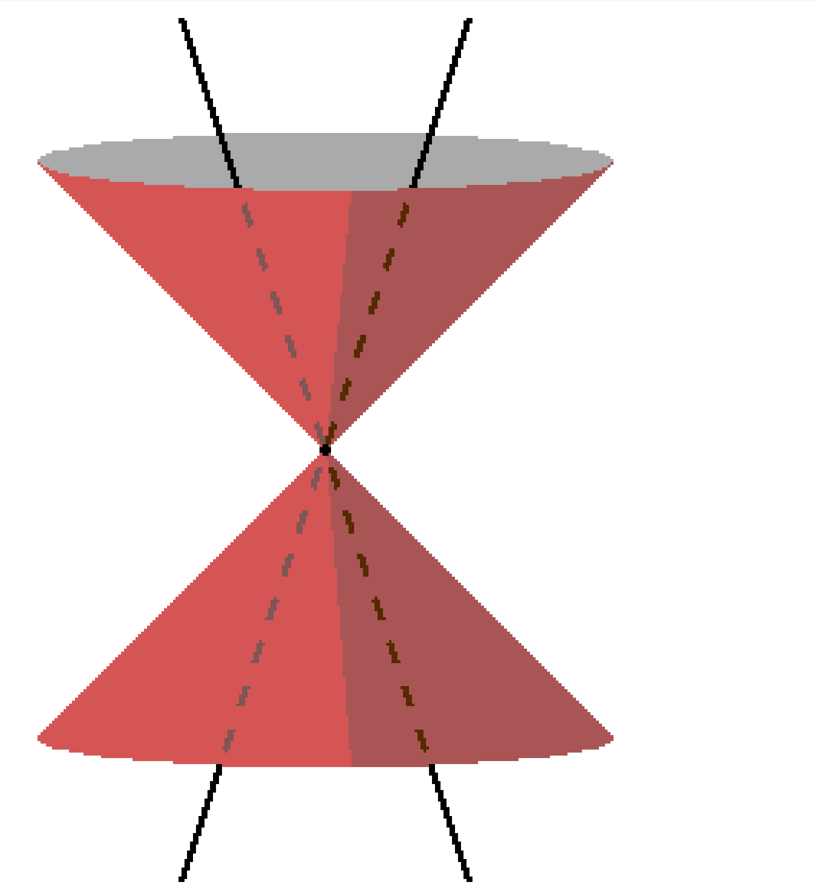
$$(\Delta s)^2 = 0: \quad \text{"light-like separation of events"}$$

$$(\Delta s)^2 < 0: \quad \text{"time-like separation of events"}$$



Signals from **O** can reach **A** and **B** but not **C**, therefore a causal relation is possible only between events **A** and **O** or between **B** and **O** but not between **C** and **O**

If the world line on which events with light-like separation from the origin lie, is rotated about the t axis, then it forms the mantel of a ***cone***. Extended to four dimensions (three spatial dimension and time), this becomes the ***light cone***.



Shown in the figure is a 2+1 dimensional image of the light cone.

We cannot draw anything in the 3+1 dimensional world, not even imagine it!

Space and time together form a ***four dimensional space-time*** with ***non-Euclidean geometry*** (Minkowski).

Meaning of *Euclidean* geometry:

a space is said to have Euclidean geometry if the length of a vector is invariant under rotations of the coordinate system.

consider the vector $\vec{r} = (x, y, z)$

its length is given by $\ell^2 \equiv |\vec{r}|^2 = x^2 + y^2 + z^2$

a rotation in the (x,y) plane is described by

$$\begin{aligned} x \to x' &= x\cos\alpha - y\sin\alpha \\ y \to y' &= x\sin\alpha + y\cos\alpha \\ z \to z' &= z \end{aligned} \qquad \textbf{(A)}$$

hence

$$\ell^2 \rightarrow \ell'^2 = x'^2 + y'^2 + z'^2$$
$$= \left(x\cos\alpha - y\sin\alpha\right)^2 + \left(x\sin\alpha + y\cos\alpha\right)^2 + z^2$$
$$= x^2\left(\cos^2\alpha + \sin^2\alpha\right) + y^2\left(\sin^2\alpha + \cos^2\alpha\right) + z^2$$
$$= \ell^2$$

The transformation **(A)** is a particular case of an orthogonal transformation.

To write it in a general form it is convenient to use the following notation:

$$x_1 = x,\ \ x_2 = y,\ \ x_3 = z$$

Then we can write a linear transformation as

$$x_i \rightarrow x_i' = \sum_{j=1}^{3} a_{ij} x_j \qquad \textbf{(B)}$$

hence

$$\ell^2 = x_1^2 + x_2^2 + x_3^2 = \sum_{i=1}^{3} x_i^2$$

and under the linear transformation **(B)** we get

$$\ell^2 \to \ell'^2 = \sum_{i=1}^{3} x_i'^2 = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} a_{ij} x_j\right)^2$$

$$= \sum_{i=1}^{3}\left(\sum_{j=1}^{3} a_{ij} x_j\right)\left(\sum_{k=1}^{3} a_{ik} x_k\right) = \sum_{j,k=1}^{3}\left(\sum_{i=1}^{3} a_{ij} a_{ik}\right) x_j x_k$$

and the requirement of invariance of the length of the vector is

$$\sum_{i=1}^{3} a_{ij} a_{ik} = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$ (Kronecker symbol).

In Newtonian mechanics the three-dimensional space is assumed to be Euclidean and time is a separate entity (*cf.* Galilean transformation!).

In relativity space and time are ***mixed*** by the Lorentz transformation, and we have seen that the Lorentz invariant square of length of a space-time interval is of the form of

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$$

The minus sign, which is a physical requirement, makes the four dimensional space non-Euclidean.

Minkowski introduced an ***imaginary time:*** $x_4 = ict$

and then the square of the interval is of the following form:

$$(\Delta s)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + (\Delta x_4)^2$$

which formally is the square of a Euclidean interval.

Thus it must be invariant under a rotation, for instance in the (x_1, x_4) plane:

$$
\begin{aligned}
x_1 &\to x_1' = x_1 \cos\alpha - x_4 \sin\alpha \\
x_4 &\to x_4' = x_1 \sin\alpha + x_4 \cos\alpha \\
x_2 &\to x_2' = x_2, \quad x_3 \to x_3' = x_3,
\end{aligned}
$$

To make contact with the real world we substitute $x_4 = ict$:

$$
\begin{aligned}
x_1 &\to x_1' = x_1 \cos\alpha - ict \sin\alpha \\
ict &\to ict' = x_1 \sin\alpha + ict \cos\alpha
\end{aligned}
$$

and use the identities

$$
i \sin\alpha = \sinh(i\alpha); \quad \cos\alpha = \cosh(i\alpha)
$$

hence

$$
\begin{aligned}
x_1 &\to x_1' = x_1 \cosh(i\alpha) - ct \sinh(i\alpha) \\
ct &\to ct' = -x_1 \sinh(i\alpha) + ct \cosh(i\alpha)
\end{aligned}
$$

and for this to be real we must put $i\alpha = y$

(this y not to be confused with the y coordinate!)

hence

$$x_1 \to x_1' = x_1 \cosh y - ct \sinh y$$
$$ct \to ct' = -x_1 \sinh y + ct \cosh y$$

Compare this with the previous form of the Lorentz transformation:

$$x \to x' = \gamma\left(x + Vt\right), \quad ct \to ct' = \gamma\left(ct + \frac{V}{c}x\right),$$

hence

$$\cosh y = \gamma; \quad \sinh y = \left(V/c\right)\gamma$$

Expressing the hyperbolic functions in terms of exponentials we get

$$\gamma = \frac{1}{2}\left(e^{y} + e^{-y}\right);$$

$$\left(V/c\right)\gamma = \frac{1}{2}\left(e^{y} - e^{-y}\right)$$

hence adding these equations:

$$e^y = \left(1 + \frac{V}{c}\right)\gamma = \sqrt{\frac{1 + V/c}{1 - V/c}}$$

and finally

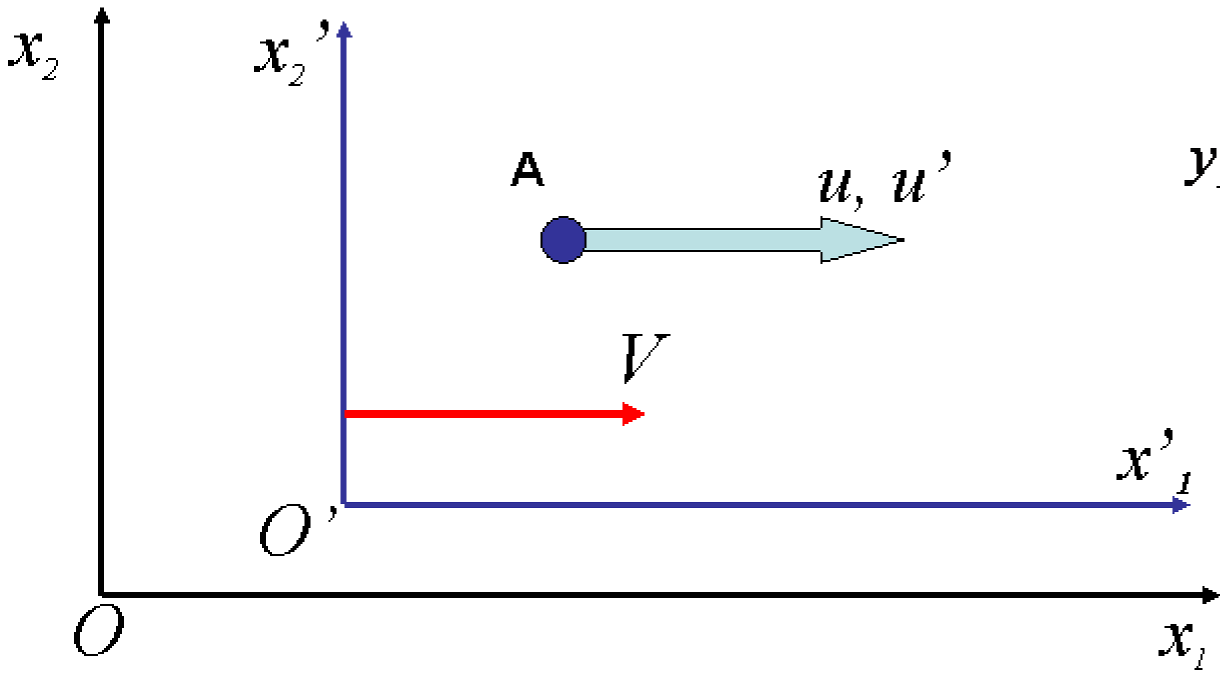
$$\boxed{y = \frac{1}{2}\ln\frac{1 + V/c}{1 - V/c}}$$

The quantity y is called the ***rapidity***

The rapidity has an interesting property: ***it is additive***!
This means that if you carry out a LT from **K** to **K'**, and then a second LT from **K'** to **K"**, then their corresponding rapidities simply add.

Addition of rapidities

$$y_A = \frac{1}{2}\ln\frac{1+u/c}{1-u/c}$$



$$y'_A = \frac{1}{2}\ln\frac{1+u'/c}{1-u'/c}$$

$$u' = \frac{u-V}{1-uV/c^2}$$

$$y'_A = \frac{1}{2}\ln\frac{1+(u-V)/(c-uV/c)}{1-(u-V)/(c-uV/c)}$$

$$= \ldots = y_A - y$$

where $$y = \frac{1}{2}\ln\frac{1+V/c}{1-V/c}$$

Thus in relativistic kinematics the ***rapidities are additive***, and not the velocities.

One should ask the question: is this consistent with the law of addition of velocities at V/c <<1?

We have

$$y_{|V/c\ll 1} = \frac{1}{2}\ln\left(\frac{1-V/c}{1+V/c}\right)_{|V/c\ll 1} = \frac{V}{c} + O\left(\frac{V^2}{c^2}\right)$$

and obviously the addition of rapidities at nonrelativistic speeds turns into the nonrelativistic addition of velocities.

Four dimensional vector space.

What Minkowski in fact realised was that the space-time continuum forms a four dimensional vector space.

This space is non-Euclidean. There are several equivalent ways of describing vectors in this 4D space:

$$\mathrm{x} = \left(x_1, x_2, x_3, x_4\right), \text{ where } \quad x_4 = ict$$

This was Minkowski's choice.

Alternatively one writes

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

these are called the ***covariant components*** of the four-vector x.

A second set of components is introduced: the ***contravariant components*** of x:

$$x_0 = ct, \quad x_1 = -x, \quad x_2 = -y, \quad x_3 = -z$$

The Lorentz invariant square of the four-vector is

$$\mathrm{x}^2 = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3$$
$$= (ct)^2 - x^2 - y^2 - z^2$$

The connection between covariant and contravariant components is established by the metric tensor

$$g_{\mu\nu} = \begin{cases} 1 & \text{if} \quad \mu = \nu = 0 \\ -1 & \text{if} \quad \mu = \nu = 1,2,3 \\ 0 & \text{if} \quad \mu \neq \nu \end{cases}$$

hence

$$x_\mu = \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu$$

Sums of products with equal upper and lower index occur frequently, such products without summation occur only very exceptionally. Therefore one uses the Einstein summation convention:

One omits the sign of summation and simply writes

$$x_\mu = g_{\mu\nu} x^\nu$$

understanding by implication that a summation must be carried out over the repeated index (in the present example the index *v*).

Care must be taken to ensure that any index that is ***not*** summed over appears on both sides of the equation ("***conservation of indices***"!).

Thus we can write the Lorentz invariant square of x as

$$\mathrm{x}^2 = x_\mu x^\mu$$

Given two 4-vectors x and y, their Lorentz invariant scalar product is

$$\mathrm{x} \cdot \mathrm{y} = x_\mu y^\mu$$

Some authors use a metric tensor whose components have the opposite signs to the ones I have written. So one must take care to know which convention is used, and certainly not mix formulas written in different conventions!

Recall that in the Minkowski metric, i.e. with $x_4 = ict$, the square of a ***time-like*** vector was ***negative*** and the square of a ***space-like*** vector was ***positive***.

In the convention of $g_{00}=1$, $g_{11}=g_{22}=g_{33}= -1$ we have the opposite:
the square of a ***time-like*** vector is ***positive*** and
the square of a ***space-like*** vector is ***negative.***

From now on I will consistently use the “+ - - -” metric.

Doppler Effect.

A plane electromagnetic wave travelling in x direction is represented by the wave function

$$u(x,t) = Ae^{i(kx-\nu t)}$$

where

$$k = 1/\lambda \quad (\text{wave number}), \nu = \text{frequency}$$

and

$$\varphi(x,t) = kx - \nu t$$

is the ***phase*** of the wave.

The phase is dimensionless. At a particular point (x,t) in space-time it is just a number. It is therefore intuitively plausible that the phase is Lorentz invariant or, in other words, is a ***scalar*** in the 4D space-time.

We can now appeal to the ***quotient theorem*** of tensor algebra.
In the particular case under consideration this theorem is the following:

If u^μ , μ=0,1,2,3, is a vector and S is a scalar, and if

$$S = v_\mu u^\mu$$

then v^μ is also a vector

The phase φ is of such a form:

$$\varphi(x,t) = kx - \nu t = -(ct, x, 0, 0)\cdot(\nu/c, k, 0, 0)$$

therefore we must conclude that

$$\mathrm{k} = (\nu/c, k, 0, 0)$$

is a four-vector. This four-vector …

must transform under Lorentz transformations like the space-time four-vector:

$$k^1 \to k^{1\prime} = \gamma\left(k^1 - \frac{V}{c}k^0\right),$$

$$k^2 \to k^{2\prime} = k^2,$$

$$k^3 \to k^{3\prime} = k^3,$$

$$k^0 \to k^{0\prime} = \gamma\left(k^0 - \frac{V}{c}k^1\right)$$

$\Longleftrightarrow$

$$k_x \to k_x{}' = \gamma\left(k_x - \frac{V}{c^2}\nu\right),$$

$$k_y \to k_y{}' = k_y,$$

$$k_z \to k_z{}' = k_z,$$

$$\nu \to \nu' = \gamma\left(\nu - Vk_x\right)$$

Now the square of the four-vector

$$\mathrm{k} = \left(\nu/c, k, 0, 0\right)$$

is equal to zero. This follows from the wave equation of electromagnetic waves:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

hence, substituting the plane wave function, we get

$$\nu^2 = c^2 k^2$$

or since all the quantities in this equation are positive, we have

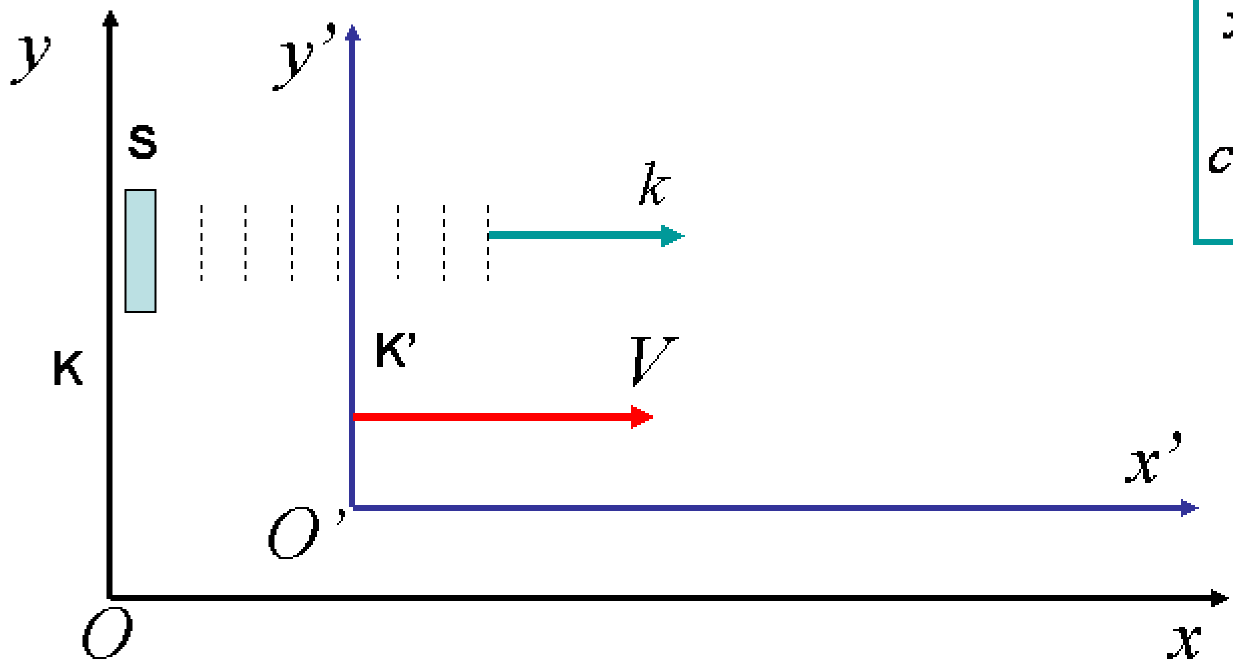
$$\nu = ck$$

Therefore

$$\nu \to \nu' = \gamma(\nu - Vk) = \gamma\left(1 - \frac{V}{c}\right)\nu = \sqrt{\frac{1 - V/c}{1 + V/c}}\,\nu$$

This is known as the ***Doppler effect***.

Doppler Effect



$$x' = \gamma(x - Vt);$$

$$ct' = \gamma\left(ct - \frac{V}{c}x\right)$$

Plane waves travelling in x direction; inertial frame **K'** is moving to the right, i.e. heading away from the source **S**.

$$\nu' = \nu\sqrt{(1 - V/c)/(1 + V/c)}$$

Thus the frequency observed if the source is moving away from the observer is less than the emitted frequency. This is seen in distant galaxies.

Compared are series of spectral lines of the light from a galaxy with known series of spectral lines measured in the laboratory. Observed is an overall shift towards smaller frequencies and hence greater wavelengths. This is called ***red shift***.

The red shift of the spectra from distant galaxies is evidence of the ***expansion of the universe***, or at least of the visible part of the universe.

The most famous formula

$$E = mc^2$$

There are many ways to derive this formula. I shall appeal to the particle-wave duality of matter.

Particle-wave duality was first established by Einstein for electromagnetic waves. He showed that a number of difficulties could be removed if it was assumed that energy and momentum of light is localised in space and not distributed over the entire wave, as required by Maxwell's theory.

One of the difficulties that had arisen was connected with the **photoelectric effect**. It was observed that the energy of the emitted electrons did depend on the frequency of the light shone on a metal surface, and that the intensity of the light was responsible only for the number of emitted electrons.

A striking confirmation of the corpuscular nature of light was the discovery of the ***Compton effect***, i.e. the reduction of the frequency of light scattered from a free electron.

The formal expression of particle-wave duality is the relation between frequency and wave length (*wave properties*!) on the one hand and energy and momentum (*particle properties*!) on the other:

$$E = h\nu; \quad \vec{p} = h\vec{k}$$

Then, if we recall that frequency and wave vector transform like a four-vector, we must conclude that energy and momentum also constitute a four-vector.

For dimensional reasons the energy-momentum four-vector is of the form

$$\mathrm{p} = \left(E/c \, , \vec{p} \right)$$

and its invariant square is

$$\mathrm{p}^2 = E^2 / c^2 - \vec{p}^{\,2} = \text{invariant}$$

We must now establish the physical meaning of the invariant.

But first it must be said that de Broglie showed on theoretical grounds that the electron also had wave properties as well as particle properties.

Independent of the work of de Broglie, Davisson and Germer discovered experimentally that a beam of electrons, reflected from a crystal, produced an interference pattern similar to the interference pattern seen when x-rays are reflected from a crystal.

Since the energy of the electrons was known and the analysis of the interference pattern gave the wave length, one could compare these quantities. The result was as postulated by de Broglie:

$$E = h\nu; \quad \vec{p} = h\vec{k}$$

Thus if we ask for the physical meaning of the invariant square of the energy-momentum four-vector, then we will put this question not only, and not so much, in the context of electromagnetic waves but mainly with matter waves in mind.

Thus we have a 4-vector

$$\mathrm{p} = \left(E/c \, , \vec{p} \right)$$

whose invariant square is

$$\mathrm{p}^2 = E^2/c^2 - \vec{p}^2 = \left(mc \right)^2$$

where I have chosen the symbol mc for the invariant because it will turn out that m is the mass of the particle.

But it must be clearly understood that at this stage we only know that this is an invariant and that it has the dimension of mass.

Now assume that we have two inertial frames K and K' with the familiar convention that they coincide at time t = 0 and that K' moves with relative velocity V in the x direction.

Also assume that the particle is moving in the x direction, i.e.

$$\vec{p} = \left(p, 0, 0 \right)$$

Then the Lorentz transformation is

$$p \rightarrow p' = \gamma\left(p - \frac{V}{c^2}E\right),$$

$$E \rightarrow E' = \gamma\left(E - Vp\right)$$

Let us choose the frame K' to be the rest frame of the particle, hence

$$p' = 0, \quad \text{and} \quad E' = mc^2$$

also

$$p = \frac{V}{c^2}E, \quad \text{hence} \quad E' = \gamma\left(1 - \frac{V^2}{c^2}\right)E = \frac{1}{\gamma}E$$

thus finally

$$\boxed{E = \gamma mc^2 \quad \text{and} \quad p = \gamma mV}$$

What is still lacking is the physical interpretation of the invariant m.

To find this consider the momentum in the nonrelativistic limit:

$$p_{|V/c\ll 1} = \gamma m V_{|V/c\ll 1} = mV\left(1+\frac{V^2}{2c^2}+\dots\right)$$

and we recover the nonrelativistic definition of momentum if we identify m with the mass of the particle.

For the particle at rest, the relativistic γ factor is equal to 1, and hence the relativistic ***rest energy*** of the particle is

$$E = mc^2$$

If the particle moves with a nonrelativistic velocity V, then we have

$$E = \gamma mc^2 = mc^2\left(1+\frac{V^2}{2c^2}+\dots\right) = mc^2 + \frac{1}{2}mV^2 + \dots$$

hence, ignoring terms of higher order of smallness, we see that the total relativistic energy is the sum of the rest energy and the nonrelativistic kinetic energy of the particle.

At higher particle velocities one therefore defines the ***relativistic K.E. T:***

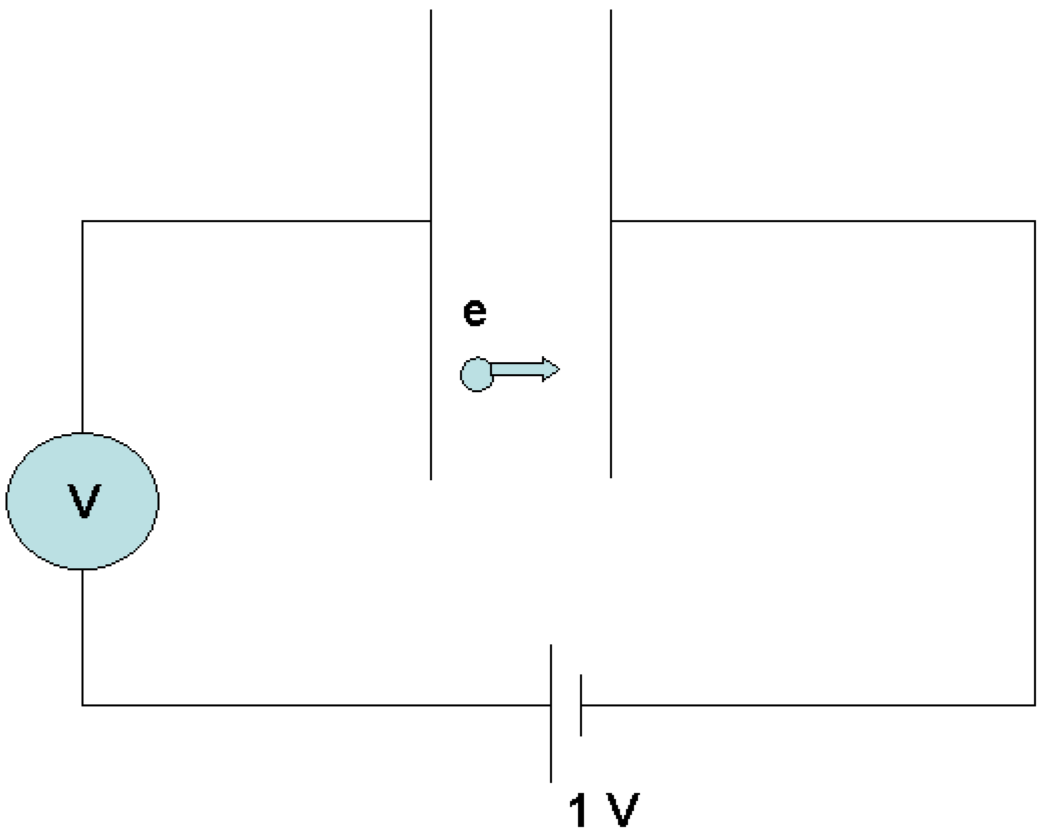
$$T = E - mc^2$$

Relativistic effects must be taken into account in atomic, nuclear and elementary particle physics. There the units like kilogram for mass and Joule for energy are inconvenient. For example, the mass of and electron is

$$m_e = 9.11 \times 10^{-31} \text{ kg}$$

Therefore one uses other units which avoid large powers of 10. The convenient unit of energy is the electron-volt. This is defined in the following way:

Units: definition of electron-Volt (eV)



The electron e acquires a K.E. of 1 eV in falling through a p.d. of 1 V

Derived units are the kilo electron-volt, Mega electron-volt, etc.

$$1\,\text{keV} = 1000\ \text{eV}$$
$$1\,\text{MeV} = 10^{6}\ \text{eV}$$
$$1\,\text{GeV} = 10^{9}\ \text{eV}$$
$$1\,\text{TeV} = 10^{12}\ \text{eV}$$

(G = Giga, T = Tera).

The rest energy of the proton is about 940 MeV, and hence its mass is 940 MeV/c^2

The highest energy accelerator to date is the **Tevatron** at the Fermilab where protons are accelerated close to 1000 GeV, *i.e.* to over 1000 times their rest energy.
At the European laboratory CERN an accelerator is under construction in which protons will be accelerated to 7 TeV.

Relativistic Mechanics:

Charged particle in a magnetic field

Particle of mass m and charge q in a magnetic field B:

The equation of motion is

$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B}$$

where

$$\vec{p} = \gamma m \vec{v}$$

and

$$\vec{F} = q\vec{v} \times \vec{B}$$ Lorentz force

For simplicity of notation write B for qB, and restore q at the end.

Thus

$$\frac{d\vec{p}}{dt} = \vec{v} \times \vec{B} = \frac{1}{\gamma m} \vec{p} \times \vec{B}$$

hence

$$\vec{p} \cdot \frac{d\vec{p}}{dt} = 0, \quad \because \ \vec{p} \cdot \left(\vec{p} \times \vec{B} \right) = 0$$

or

$$\frac{dp^2}{dt} = 0$$

i.e.

$$p^2 = const.$$

and hence

$$E^2 = p^2c^2 + m^2c^4 = const.$$

and

$$v = \frac{p}{E} = const.$$

Consider a ***solenoidal field*** in z direction:

$$\vec{B} = B\left(0,0,1\right) \equiv B\hat{z}$$

and then consider two cases: (i) p ∥ B, and (ii) p ⊥ B.

Case (i): p ∥ B: then

$$\frac{d\vec{p}}{dt} = 0$$

hence

$$\vec{p} = const.; \quad \vec{v} = const.$$

i.e. the particle continues to travel with its initial velocity.

Case (ii): p ⊥ B:

$$\vec{p} \times \vec{B} = B\left(p_y, -p_x, 0\right)$$

hence

$$\frac{dp_z}{dt} = 0, \quad p_z = \text{const.}$$

and if initially p_z = 0, then the particle continues to move in the (x, y) plane