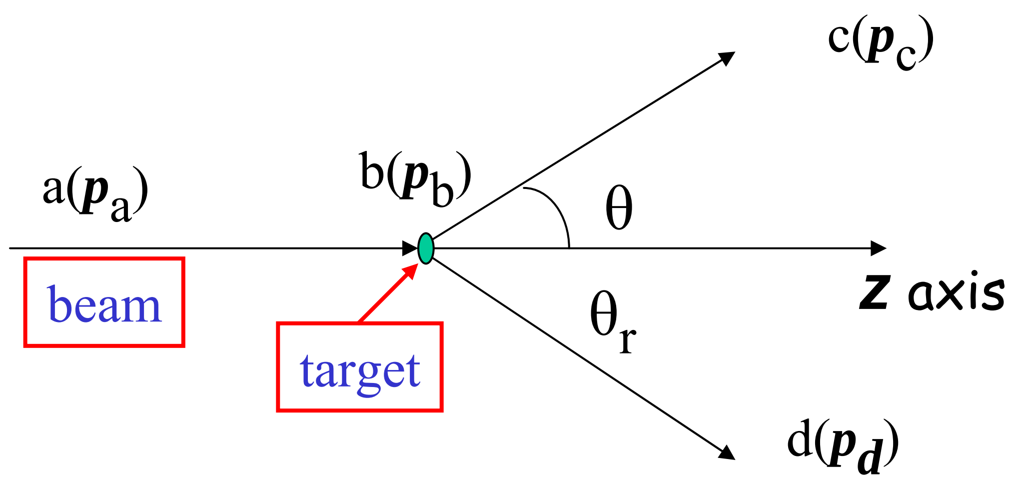
High Energy Physics

Lecture 3:

Kinematics of Particle Reactions



LAB kinematics diagram of particle collision $a + b \rightarrow c + d$

Definition of LAB frame: the target particle is at rest

The particles have masses m_a, m_b, m_c, m_d

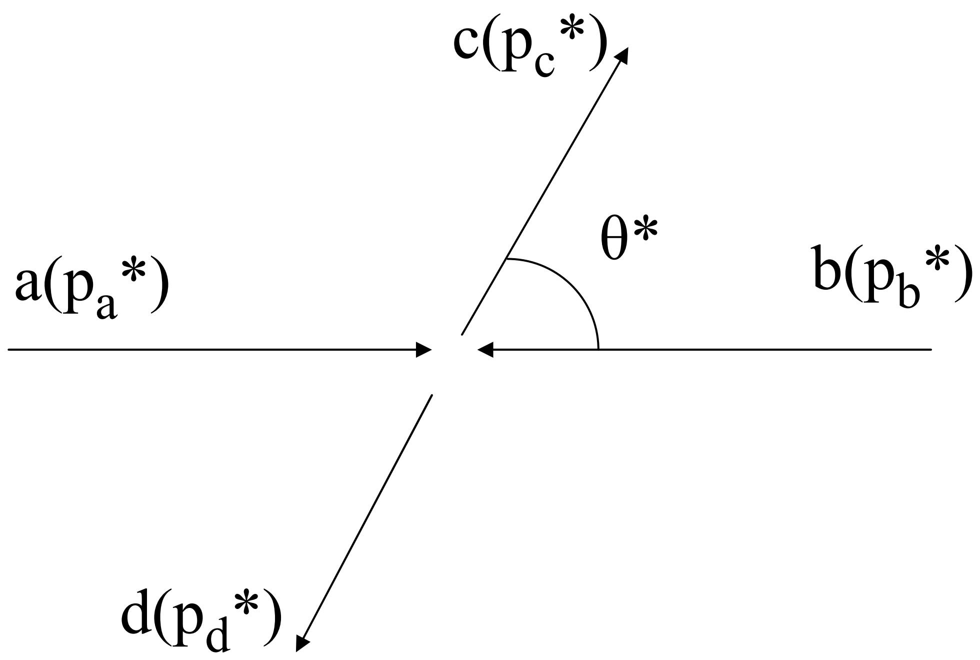
Particle a is the incident particle

Particle b is the target particle: this is initially at rest in the LAB

Particle c is the scattered particle

Particle d is the recoil particle

θ is the LAB scattering angle; θ_r is the recoil angle



Kinematics diagram of particle collision in the CMS

Definition of CMS:
the total momentum of the initial system is zero.

It follows by momentum conservation that the total momentum of the final system is also equal to zero.

Energy and momentum conservation:

$$E_a + E_b = E_c + E_d$$

$$\vec{p}_a + \vec{p}_b = \vec{p}_c + \vec{p}_d \qquad \text{(vectors!)}$$

$E = \sqrt{(mc^2)^2 + (pc)^2}$ is the total relativistic energy

c = speed of light (in the vacuum!)

In particle physics one frequently uses units such that

$$c = 1 \quad (\text{and} \quad \hbar = 1 \ )$$

and then the energy-momentum relation is

$$E = \sqrt{m^2 + p^2}$$

Elastic scattering: $m_a = m_c$, $m_b = m_d$

Typical problem of particle kinematics
(needed by experimentalists):

Given the masses of the initial particles and the momentum of the incident particle, find the momenta of the final particles in an elastic collision;

also find the LAB recoil angle.

The problem is solved by using energy and momentum conservation:

$$p_c = p\,\frac{(s + m_a^2 - m_b^2)\cos\theta \pm 2W\sqrt{m_b^2 - m_a^2 \sin^2\theta}}{2(s + p^2\sin^2\theta)}$$

where $\boldsymbol{p}$ is the LAB momentum of $\boldsymbol{a}$,

$W = E_{LAB} + m_b$ is the total LAB energy of the initial system,

$$s = m_a^2 + m_b^2 + 2m_b E_{LAB}, \qquad p_d = \sqrt{p^2 + p_c^2 - 2pp_c\cos\theta}$$

$$\tan\theta_r = p_c \sin\theta / (p - p_c\cos\theta)$$

Example: elastic pion – proton scattering

m_π =140 MeV, m_p=940 MeV,

Assume: LAB energy of the incident pion E = 1000 MeV

Let θ = 30 deg

Question: what are the momenta and energies of the scattered pion and of the recoil proton; what is the recoil angle?

Using the above formulas we get the following answer:

$$p_\pi = 866.3 \text{ MeV}, \quad E_\pi = 877.54 \text{ MeV}$$
$$p_p = 495.2 \text{ MeV}, \quad E_p = 1062.46 \text{ MeV}$$
$$\theta_r = 1.07 \text{ rad} = 61 \text{ deg.}$$

Note the conservation of energy: initially we had a total energy

$W = E+m_p$ = 1940 MeV;

after the collision we have $E_\pi + E_p$ = 877.54+1062.46 = 1940 MeV

Our result looks surprising: the proton, which was initially at rest and which was hit by a pion of LAB energy 1000 MeV, has acquired an energy of 1062 MeV!

The reason is that these energies are ***relativistic total energies***: they include the rest energy, which in the case of the proton is 940 MeV.

More intuitive than the total energy is the kinetic energy (K.E.): this is defined by K.E. = total energy – rest energy:

$$T = E - mc^2$$

We can check that the expression for T takes on the familiar form of the nonrelativistic K.E. if the particle velocity is small compared with c:

$$T = \sqrt{m^2c^4 + p^2c^2} - mc^2 = mc^2\sqrt{1 + p^2/m^2c^2} - mc^2$$

and since nonrelativistically $p/mc << 1$ we have

$$\sqrt{1 + p^2/m^2c^2} \approx 1 + p^2/2m^2c^2 \qquad \text{hence} \qquad T_{nr} \approx \frac{p^2}{2m}$$

In our example of elastic pion – proton scattering, the energy balance, expressed in terms of the K.E.s, is

$$(T_\pi + m_\pi c^2) + (T_p + m_p c^2) = (T_\pi^{'} + m_\pi c^2) + (T_p^{'} + m_p c^2)$$

i.e. the rest energies cancel and we are left with the balance of K.E.s:

$$T_\pi + T_p = T_\pi^{'} + T_p^{'}$$

and in our previous numerical example we have:

$$T_\pi = 860 \text{ MeV}, T_p = 0, \quad T_\pi^{'} = 737.54 \text{ MeV}, T_p^{'} = 112.46 \text{ MeV}$$

and we see that only a small part of the initial pion K.E. got transferred to the proton.

But remember: the K.E.s balance only in the case of elastic scattering. In ***inelastic*** collisions only the ***total*** energies balance!

Inelastic collisions.

Example of an inelastic collision:

$$\pi^+ + p \to \pi^+ + p + \pi^0$$

Here an additional (neutral) pion has been created.
A first question is: what LAB K.E. is needed to produce this extra pion?
(The smallest energy to produce the extra pion is called ***threshold energy***)

The calculation is simple in the CMS.
To find the LAB threshold K.E. we then have two ways:

1) do a Lorentz transformation from CMS to LAB; this is difficult

or

2) use the concept of invariance: that's the easy way

Let us do the calculation in the CMS, then use invariance.

CMS threshold energy.

By definition of the CMS, the total momentum is equal to zero, both before and after the reaction.

Before the reaction we have

$$\vec{p}_\pi + \vec{p}_p = 0, \quad \text{hence} \quad p_\pi = p_p$$

and we can drop the subscripts on the momenta; the total CMS energy before the collision is therefore

$$E_{in} = \sqrt{m_\pi^2 + p^2} + \sqrt{m_p^2 + p^2}$$

("*in*" for initial)

Usually the square of E_{in} is denoted by ***s***, *i.e.* we have

$$\sqrt{s} = \sqrt{m_\pi^2 + p^2} + \sqrt{m_p^2 + p^2}$$

and if we solve for ***p*** (Exercise!), then we get

$$p_{CMS} = \sqrt{\left[s - \left(m_p - m_\pi\right)^2\right]\left[s - \left(m_p + m_\pi\right)^2\right]} \Big/ 2\sqrt{s}$$

After the reaction we have in the CMS

$$s = \left\{\sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_2^2} + \sqrt{m_3^2 + p_3^2}\right\}^2$$

and the minimum of this corresponds to ***all*** final state particles being at rest in the CMS, *i.e.* $\boldsymbol{p}_{\mathrm{i}} = 0$, i=1,2,3, hence

$$s_{\min} = M^2 = \{m_1 + m_2 + m_3\}^2$$

One can show that the quantity ***s*** is given in terms of the LAB energy of the incident particle by the following expression:

$$s = m_\pi^2 + m_p^2 + 2m_p E_{LAB}$$

and this is ***invariant***, meaning that it takes the same value in any reference frame. We can therefore equate the threshold value of ***s*** in the LAB with its threshold value in the CMS, hence

$$E_{LAB}^{thr} = \frac{1}{2m_p}\left(M^2 - m_\pi^2 - m_p^2\right)$$

Putting in the numbers for our example of production of a neutral pion in a pion-proton collision:

$$\pi^+ + p \rightarrow \pi^+ + p + \pi^0$$

$$m_{\pi^+} = 140\,\text{MeV},\ m_{\pi^0} = 135\ \text{MeV},\ m_p = 940\ \text{MeV}$$

we get (Exercise!): $M = 1215\ \text{MeV}$

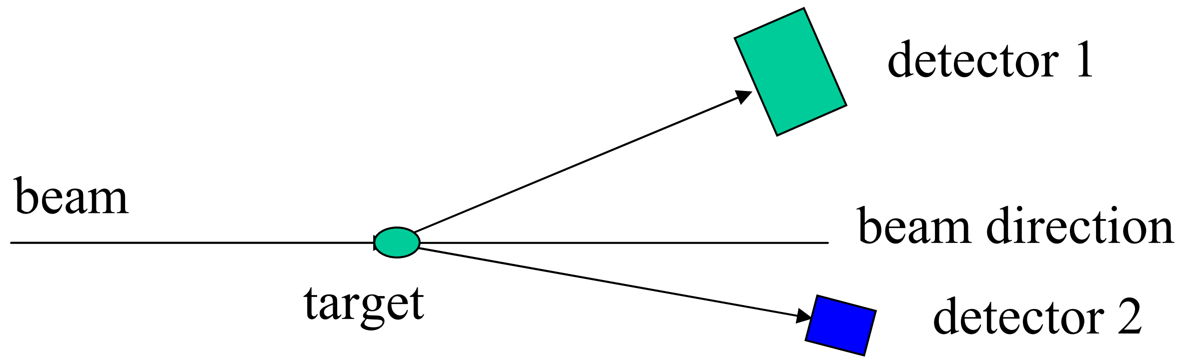
and for the LAB energy and LAB K.E. of the incident pion (Exercise!)

$$E_{LAB}^{thr} = 305\ \text{MeV}, \qquad T_{LAB}^{thr} = 165\ \text{MeV}$$

Thus the LAB threshold K.E. is greater than the extra mass produced. This is so because in the LAB the entire final state system is moving, *i.e.* it has a K.E. which is taken from the K.E. of the incident particle.

Use of kinematics in planning an experiment:

In a 2-to-2 reaction, the final momenta span a plane, called the reaction plane. By momentum conservation the reaction plane contains the beam axis (**z** axis!). We can therefore position two detectors such as to detect the two final state particles:



We can do something a bit more clever than just position the detectors at the angles expected for the reaction we want to study: we know also the speed of the particles and therefore the time it takes them to get to the detectors: *Time of Flight* (ToF). So we put the detectors at such distances that the ToFs coincide, then use electronics to count only particles which arrive at the detectors in coincidence.

Time of Flight calculation

Distance between interaction point (target!) and detector: D
Particle velocity ……………………………………………...: v
hence ToF ………………………………………………...: t = D/v

But velocities are relativistic in typical particle physics experiments!

The relativistic relation between momentum and velocity is

$$p = \gamma m v$$

where $\gamma = 1 \Big/ \sqrt{1-(v/c)^2}$ ("relativistic γ factor")

The relativistic relation for the energy is

$$E = \gamma m c^2$$

hence $v/c = pc/E$

this is the particle velocity "in units of c"; one says for instance:
"*the particle is travelling at a speed of 0.9 of the speed of light*".

Example: $\pi^+ p \longrightarrow \pi^+ p$ with $E_{LAB} = 1000$ MeV

Note that we have plenty of energy to produce additional pions: to produce just a single (neutral) pion we need only 305 MeV; you can work out for yourselves how many pions can be produced if the beam energy is 1000 MeV (Exercise!).

So the design of our experiment should be such as to count all elastic collisions and not to count the inelastic ones.

We have previously worked out for a scattering angle of 30 deg:

$p_\pi = 866.3$ MeV/c; $E_\pi = 877.54$ MeV
$p_p = 495.2$ MeV/c; $E_p = 1062.46$ MeV and $\theta_r = 61$ deg.

Hence $v_\pi = 866.3/877.54 = 0.987\,c$ and $v_p = 0.466\,c$

If we put the pion detector at a distance $D_\pi = 1$ m from the target, then the ToF of the pion is

$$t_\pi = \frac{1m}{0.987c} \approx 0.34ns$$

For coincidence the proton ToF must be equal to the pion ToF; therefore the distance at which we must put the proton detector is

$$D_p = v_p \cdot t_\pi = D_\pi \cdot \frac{v_p}{v_\pi} = \frac{0.466}{0.987} = 0.47m$$

Please remember to do all indicated exercises; also check my numbers: I may have had butterfingers and got the numbers wrong!